

# EFFICIENT PRECONDITIONERS FOR SADDLE POINT SYSTEMS WITH TRACE CONSTRAINTS COUPLING 2D AND 1D DOMAINS \*

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**Abstract.** We study preconditioners for a model problem describing the coupling of two elliptic subproblems posed over domains with different topological dimension by a parameter dependent constraint. A pair of parameter robust and efficient preconditioners is proposed and analyzed. Robustness and efficiency of the preconditioners is demonstrated by numerical experiments.

**Key words.** preconditioning, saddle-point problem, Lagrange multipliers

**AMS subject classifications.** 65F08

**1. Introduction.** This paper is concerned with preconditioning of multiphysics problems where two subproblems of different dimensionality are coupled. We assume that  $\Gamma$  is a sub-manifold contained within  $\Omega \in \mathbb{R}^n$  and consider the following problem:

$$-\Delta u + \epsilon \delta_\Gamma p = f \quad \text{in } \Omega, \quad (1.1a)$$

$$-\Delta v - p = g \quad \text{on } \Gamma, \quad (1.1b)$$

$$\epsilon u - v = 0 \quad \text{on } \Gamma, \quad (1.1c)$$

where  $\delta_\Gamma$  is a function with properties similar to the Dirac delta function as will be discussed later. To allow for a unique solution  $(u, v, p)$  the system must be equipped with suitable boundary conditions and we shall here, for simplicity, consider homogeneous Dirichlet boundary conditions for  $u$  and  $v$  on  $\partial\Omega$  and  $\partial\Gamma$  respectively. We note that the unknowns  $u, v$  are here the primary variables, while the unknown  $p$  should be interpreted as a Lagrange multiplier associated with the constraint (1.1c).

The two elliptic equations that are stated on two different domains,  $\Omega$  and  $\Gamma$ , are coupled and therefore the restriction of  $u$  to  $\Gamma$  and the extension of  $p$  to  $\Omega$  are crucial. When the codimension of  $\Gamma$  is one, the restriction operator is a trace operator and the extension operator is similar to the Dirac delta function. We note that  $\epsilon \in (0, 1)$  and that the typical scenario will be that  $\epsilon \ll 1$ . We will therefore focus on methods that are robust in  $\epsilon$ .

The problem (1.1a)–(1.1c) is relevant to biomedical applications [18, 15, 2, 17] where it models the coupling of the porous media flow inside tissue to the vascular bed through Starlings law. Further, problems involving coupling of the finite element method and the boundary element method, e.g. [24, 26], are of the form (1.1). The system is also relevant for domain decomposition methods based on Lagrange multipliers [32]. Finally, in solid mechanics, the problem of plates reinforced with ribs, cf. for example [44, ch. 9.11], can be recast into a related fourth order problem. We also

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note that the techniques developed here to address the constraint (1.1c) are applicable in preconditioning fluid-structure interaction problems involving interactions with thin structures, e.g. filaments [22].

One way of deriving equations (1.1) is to consider the following minimization problem

$$\left. \begin{aligned} \int_{\Omega} (\nabla u)^2 - 2uf \, dx \\ \int_{\Gamma} (\nabla v)^2 - 2vg \, ds \end{aligned} \right\} \rightarrow \min \quad (1.2)$$

subject to the constraint

$$\epsilon u - v = 0 \quad \text{on } \Gamma. \quad (1.3a)$$

Using the method of Lagrange multipliers, the constrained minimization problem will be re-cast as a saddle-point problem. The saddle-point problem is then analyzed in terms of the Brezzi conditions [13] and efficient solution algorithms are obtained using operator preconditioning [35]. A main challenge is the fact that the constraint (1.3a) necessitates the use of trace operators which leads to operators in fractional Sobolev spaces on  $\Gamma$ .

An outline of the paper is as follows: Section 2 presents the necessary notation and mathematical framework needed for the analysis. Then the mathematical analysis as well as the numerical experiments of two different preconditioners are presented in §3 and §4, respectively. Section 5 discusses computational efficiency of both methods.

**2. Preliminaries.** Let  $X$  be a Hilbert space of functions defined on a domain  $D$  and let  $\|\cdot\|_X$  denote its norm. The  $L^2$  inner product on a domain  $D$  is denoted  $(\cdot, \cdot)_D$  or  $\int_D \cdot$ , while  $\langle \cdot, \cdot \rangle_D$  denotes the corresponding duality pairing between a Hilbert space  $X$  and its dual space  $X^*$ . We will use  $H^m = H^m(D)$  to denote the Sobolev space of functions on  $D$  with  $m$  derivatives in  $L^2 = L^2(D)$ . The corresponding norm is denoted  $\|\cdot\|_{m,D}$ . In general, we will use  $H_0^m$  to denote the closure in  $H^m$  of the space of smooth functions with compact support in  $D$  and seminorm is denoted as  $|\cdot|_{m,D}$ .

The space of bounded linear operators mapping elements of  $X$  to  $Y$  is denoted  $\mathcal{L}(X, Y)$  and if  $Y = X$  we simply write  $\mathcal{L}(X)$  instead of  $\mathcal{L}(X, X)$ . If  $X$  and  $Y$  are Hilbert spaces, both continuously contained in some larger Hilbert space, then the intersection  $X \cap Y$  and the sum  $X + Y$  are both Hilbert spaces with norms given by

$$\|x\|_{X \cap Y}^2 = \|x\|_X^2 + \|x\|_Y^2 \quad \text{and} \quad \|z\|_{X+Y}^2 = \inf_{\substack{x \in X, y \in Y \\ z = x+y}} (\|x\|_X^2 + \|y\|_Y^2).$$

In the following  $\Omega \subset \mathbb{R}^n$  is an open connected domain with Lipschitz boundary  $\partial\Omega$ . The trace operator  $T$  is defined by  $Tu = u|_{\Gamma}$  for  $u \in C(\overline{\Omega})$  and  $\Gamma$  a Lipschitz submanifold of codimension one in  $\Omega$ . The trace operator extends to bounded and surjective linear operator  $T : H^1(\Omega) \rightarrow H^{\frac{1}{2}}(\Gamma)$ , see e.g. [1, ch. 7]. The fractional Sobolev space  $H^{\frac{1}{2}}(\Gamma)$  can be equipped with the norm

$$\|u\|_{H^{\frac{1}{2}}(\Gamma)}^2 = \|u\|_{L^2(\Gamma)}^2 + \int_{\Gamma \times \Gamma} \frac{|u(x) - u(y)|^2}{|x - y|^{n+1}} \, dx dy. \quad (2.1)$$

However, the trace is not surjective as an operator from  $H_0^1(\Omega)$  into  $H^{\frac{1}{2}}(\Gamma)$ , in particular the constant function  $1 \in H^{\frac{1}{2}}(\Gamma)$  is not in the image of the trace operator.

Note that  $H_0^{\frac{1}{2}}(\Gamma)$  does not characterize the trace space, since  $H_0^{\frac{1}{2}}(\Gamma) = H^{\frac{1}{2}}(\Gamma)$ , see [30, ch. 2, thm. 11.1]. Instead, the trace space can be identified as  $H_{00}^{\frac{1}{2}}(\Gamma)$ , defined as the subspace of  $H^{\frac{1}{2}}(\Gamma)$  for which extension by zero into  $H^{\frac{1}{2}}(\tilde{\Gamma})$  is continuous, for some suitable extension domain  $\tilde{\Gamma}$  extending  $\Gamma$  (e.g.  $\tilde{\Gamma} = \Gamma \cup \partial\Omega$ ). To be precise, the space  $H_{00}^{\frac{1}{2}}(\Gamma)$  can be characterized with the norm

$$\|u\|_{H_{00}^{\frac{1}{2}}(\Gamma)} = \|\tilde{u}\|_{H^{\frac{1}{2}}(\tilde{\Gamma})}, \quad \tilde{u}(x) = \begin{cases} u(x) & x \in \Gamma \\ 0 & x \notin \Gamma. \end{cases} \quad (2.2)$$

The space  $H_{00}^{\frac{1}{2}}(\Gamma)$  does not depend on the extension domain  $\tilde{\Gamma}$ , since the norms induced by different choices of  $\tilde{\Gamma}$  will be equivalent.

The above norms (2.1)–(2.2) for the fractional spaces are impractical from an implementation point of view, and we will therefore consider the alternative construction following [30, ch. 2.1] and [16]. For  $u, v \in H_0^1(\Gamma)$ , set  $L_u(v) = (u, v)_{\Gamma}$ . Then  $L_u$  is a bounded linear functional on  $H_0^1(\Gamma)$  and in accordance with the Riesz–Fréchet theorem there is an operator  $S \in \mathcal{L}(H_0^1(\Gamma))$  such that

$$(Su, w)_{H_0^1(\Omega)} = L_u(w) = (u, w)_{\Gamma}, \quad u, w \in H_0^1(\Gamma). \quad (2.3)$$

The operator  $S$  is self-adjoint, positive definite, injective and compact. Therefore the spectrum of  $S$  consists of a nonincreasing sequence of positive eigenvalues  $\{\lambda_k\}_{k=1}^{\infty}$  such that  $0 < \lambda_{k+1} \leq \lambda_k$  and  $\lambda_k \rightarrow 0$ , see e.g. [48, ch. X.5, thm. 2]. The eigenvectors  $\{\phi_k\}_{k=1}^{\infty}$  of  $S$  satisfy the generalized eigenvalue problem

$$A\phi_k = \lambda_k^{-1} M\phi_k$$

where operators  $A, M$  are such that  $\langle Au, v \rangle_{\Gamma} = (\nabla u, \nabla v)_{\Gamma}$  and  $\langle Mu, v \rangle_{\Gamma} = (u, v)_{\Gamma}$ . The set of eigenvectors  $\{\phi_k\}_{k=1}^{\infty}$  forms a basis of  $H_0^1(\Gamma)$  orthogonal with respect the inner product of  $H_0^1(\Gamma)$  and orthonormal with respect to the inner product on  $L^2(\Gamma)$ . Then for  $u = \sum_k c_k \phi_k \in \text{span } \{\phi_k\}_{k=1}^{\infty}$  and  $s \in [-1, 1]$ , we set

$$\|u\|_{H_s} = \sqrt{\sum_k c_k^2 \lambda_k^{-s}} \quad (2.4)$$

and define  $H_s$  to be the closure of  $\text{span } \{\phi_k\}_{k=1}^{\infty}$  in the above norm. Then  $H_0 = L^2(\Gamma)$  and  $H_1 = H_0^1(\Gamma)$ , with equality of norms. Moreover, we have  $H_{\frac{1}{2}} = H_{00}^{\frac{1}{2}}(\Gamma)$  with equivalence of norms. This essentially follows from the fact that  $H_{\frac{1}{2}}$  and  $H_{00}^{\frac{1}{2}}(\Gamma)$  are closely related interpolation spaces, see [16, thm. 3.4]. Note that we also have  $H_{-1} = (H_0^1(\Gamma))^* = H^{-1}(\Gamma)$  and  $H_{-\frac{1}{2}} = (H_{00}^{\frac{1}{2}}(\Gamma))^* = H^{-\frac{1}{2}}(\Gamma)$ .

As the preceding paragraph suggests we shall use normal font to denote linear operators, e.g.  $A$ . To signify that the particular operator acts on a vector space with multiple components we employ calligraphic font, e.g.  $\mathcal{A}$ . Vectors and matrices are denoted by the sans serif font, e.g.,  $\mathbf{A}$  and  $\mathbf{x}$ . In case the matrix has a block structure it is typeset with the blackboard bold font, e.g.  $\mathbb{A}$ . Matrices and vectors are related to the discrete problems as follows, see also [35, ch. 6]. Let  $V_h \subset H_0^1(D)$  and let the discrete operator  $A_h : V_h \rightarrow V_h^*$  be defined in terms of the Galerkin method:

$$\langle A_h u_h, v_h \rangle_D = \langle Au, v_h \rangle_D, \quad \text{for } u_h, v_h \in V_h \text{ and } u \in H_0^1(D).$$

Let  $\psi_j, j \in [1, m]$  the basis functions of  $V_h$ . The matrix equation,

$$\mathbf{A}\mathbf{u} = \mathbf{f}, \quad \mathbf{u} \in \mathbb{R}^m \text{ and } \mathbf{f} \in \mathbb{R}^m$$

is obtained as follows: Let  $\pi_h : V_h \rightarrow \mathbb{R}^m$  and  $\mu_h : V_h^* \rightarrow \mathbb{R}^m$  be given by

$$v_h = \sum_j (\pi_h v_h)_j \psi_j, \quad v_h \in V_h \quad \text{and} \quad (\mu_h f_h)_j = \langle f_h, \psi_j \rangle_D, \quad f_h \in V_h^*.$$

Then

$$\mathbf{A} = \mu_h A_h \pi_h^{-1}, \quad \mathbf{v} = \pi_h v_h, \quad \mathbf{f} = \mu_h f_h.$$

A discrete equivalent to the  $H_s$  inner product (2.4) is constructed in the following manner, similar to the continuous case. There exists a complete set of eigenvectors  $\mathbf{u}_i \in \mathbb{R}^m$  with the property  $\mathbf{u}_j^\top \mathbf{M} \mathbf{u}_i = \delta_{ij}$  and  $m$  positive definite (not necessarily distinct) eigenvalues  $\lambda_i$  of the generalized eigenvalue problem  $\mathbf{A} \mathbf{u}_i = \lambda_i \mathbf{M} \mathbf{u}_i$ . Equivalently the matrix  $\mathbf{A}$  can be decomposed as  $\mathbf{A} = (\mathbf{M} \mathbf{U}) \Lambda (\mathbf{M} \mathbf{U})^\top$  with  $\Lambda = \text{diag}(\lambda_1, \dots, \lambda_m)$  and  $\text{col}_i \mathbf{U} = \mathbf{u}_i$  so that  $\mathbf{U}^\top \mathbf{M} \mathbf{U} = \mathbf{I}$  and  $\mathbf{U}^\top \mathbf{A} \mathbf{U} = \Lambda$ . We remark that  $\mathbf{A}$  is the stiffness matrix, while  $\mathbf{M}$  is the mass matrix.

Let now  $\mathbf{H} : \mathbb{R} \rightarrow \mathbf{P}_{\text{sym}}$ , where  $\mathbf{P}_{\text{sym}}$  denotes the space of symmetric positive definite matrices, be defined as

$$\mathbf{H}(s) = (\mathbf{M} \mathbf{U}) \Lambda^s (\mathbf{M} \mathbf{U})^\top. \quad (2.5)$$

Note that due to  $\mathbf{M}$  orthonormality of the eigenvectors the inverse of  $\mathbf{H}(s)$  is given as  $\mathbf{H}(s)^{-1} = \mathbf{U} \Lambda^{-s} \mathbf{U}^\top$ . To motivate the definition of the mapping, we shall in the following example consider several values  $\mathbf{H}(s)$  and show the relation of the matrices to different Sobolev (semi) norms of functions in  $V_h$ .

**EXAMPLE 2.1** ( $L_2$ ,  $H_0^1$  and  $H^{-1}$  norms in terms of matrices). Let  $V_h \subset H_0^1(\Gamma)$ ,  $\dim V_h = m$ ,  $v_h \in V_h$  and  $\mathbf{v} \in \mathbb{R}^m$  the representation of  $v_h$  in the basis of  $V_h$ , i.e.  $\mathbf{v} = \pi_h v_h$ . The  $L^2$  norm of  $v_h$  is given through the mass matrix  $\mathbf{M}$  as  $\|v_h\|_{0,\Gamma}^2 = \mathbf{v}^\top \mathbf{M} \mathbf{v}$  and  $\mathbf{M} = \mathbf{H}(0)$ . Similarly for the  $H_0^1$  (semi) norm it holds that  $|v_h|_{1,\Gamma}^2 = \mathbf{v}^\top \mathbf{A} \mathbf{v}$ , where  $\mathbf{A}$  is the stiffness matrix, and  $\mathbf{A} = \mathbf{H}(1)$ . Finally a less trivial example, let  $f_h \in V_h$  and consider  $f_h$  as a bounded linear functional,  $\langle f_h, v_h \rangle_\Gamma = (f_h, v_h)_\Gamma$  for  $v_h \in V_h$ . Then  $\|f_h\|_{-1,\Gamma}^2 = \mathbf{f}^\top \mathbf{H}(-1) \mathbf{f}$ . By Riesz representation theorem there exists a unique  $u_h \in V_h$  such that  $(\nabla u_h, \nabla v_h)_\Gamma = \langle f_h, v_h \rangle_\Gamma$  for all  $v_h \in V_h$  and  $\|f_h\|_{-1,\Gamma} = |u_h|_{1,\Gamma}$ . The latter equality yields  $\|f_h\|_{-1,\Gamma}^2 = \mathbf{u}^\top \mathbf{A} \mathbf{u}$  but since  $u_h \in V_h$  is given by the Riesz map, the coordinate vector comes as a unique solution of the system  $\mathbf{A} \mathbf{u} = \mathbf{M} \mathbf{f}$ , i.e.  $\mathbf{u} = \mathbf{A}^{-1} \mathbf{M} \mathbf{f}$  (see e.g. [33, ch. 3]). Thus  $\|f_h\|_{-1,\Gamma}^2 = \mathbf{f}^\top \mathbf{M} \mathbf{A}^{-1} \mathbf{M} \mathbf{f}$ . The matrix product in the expression is then  $\mathbf{H}(-1)$ .

In general let  $\mathbf{c}$  be the representation of vector  $\mathbf{u} \in \mathbb{R}^m$  in the basis of eigenvectors  $\mathbf{u}_i$ ,  $\mathbf{u} = \mathbf{U} \mathbf{c}$ . Then

$$\mathbf{u}^\top \mathbf{H}(s) \mathbf{u} = \mathbf{c}^\top \Lambda^s \mathbf{c} = \sum_j c_j^2 \lambda_j^s$$

and so  $\mathbf{u}^\top \mathbf{H}(s) \mathbf{u} = \|u_h\|_{H_s}^2$  for  $u_h \in V_h$  such that  $u_h = \pi_h^{-1} \mathbf{u}$ . Similar to the continuous case the norm can be obtained in terms of powers of an operator

$$\mathbf{u}^\top \mathbf{H}(s) \mathbf{u} = \left[ \mathbf{U} \Lambda^{\frac{s}{2}} (\mathbf{M} \mathbf{U})^\top \mathbf{u} \right]^\top \mathbf{M} \left[ \mathbf{U} \Lambda^{\frac{s}{2}} (\mathbf{M} \mathbf{U})^\top \mathbf{u} \right] = \left[ \mathbf{S}^{-\frac{s}{2}} \mathbf{u} \right]^\top \mathbf{M} \left[ \mathbf{S}^{-\frac{s}{2}} \mathbf{u} \right],$$

where  $\mathbf{S} = \mathbf{A}^{-1} \mathbf{M}$  is the matrix representation of the Riesz map  $H^{-1}(\Gamma) \rightarrow H_0^1(\Gamma)$  in the basis of  $V_h$ .

REMARK 2.1. *The norms constructed above for the discrete space are equivalent to, but not identical to the  $H_s$ -norm from the continuous case.*

Before considering proper preconditioning of the weak formulation of problem (1.1) we illustrate the use of operator preconditioning with an example of a boundary value problem where operators in fractional spaces are utilized to weakly enforce the Dirichlet boundary conditions by Lagrange multipliers [6].

EXAMPLE 2.2 (Dirichlet boundary conditions using Lagrange multiplier). *The problem considered in [6] reads: Find  $u$  such that*

$$\begin{aligned} -\Delta u + u &= f && \text{in } \Omega, \\ u &= g && \text{on } \Gamma \subset \partial\Omega, \\ \partial_n u &= 0 && \text{on } \partial\Omega \setminus \Gamma. \end{aligned} \quad (2.6)$$

*Introducing a Lagrange multiplier  $p$  for the boundary value constraint and a trace operator  $T : H^1(\Omega) \rightarrow H^{\frac{1}{2}}(\Gamma)$  leads to a variational problem for  $(u, p) \in H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$  satisfying*

$$\begin{aligned} (\nabla u, \nabla v)_\Omega + (u, v)_\Omega + \langle p, Tv \rangle_\Gamma &= (f, v)_\Omega && v \in H^1(\Omega), \\ \langle q, Tu \rangle_\Gamma &= \langle q, g \rangle_\Gamma && q \in H^{-\frac{1}{2}}(\Gamma). \end{aligned} \quad (2.7)$$

*In terms of the framework of operator preconditioning, the variational problem (2.7) defines an equation*

$$\mathcal{A}x = b, \quad \text{where} \quad \mathcal{A} = \begin{bmatrix} -\Delta_\Omega + I & T' \\ T & 0 \end{bmatrix}. \quad (2.8)$$

*In [6] the problem is proved to be well-posed and therefore  $\mathcal{A} : V \rightarrow V^*$  is a symmetric isomorphism, where  $V = H^1(\Omega) \times H^{-\frac{1}{2}}(\Gamma)$  and  $x \in V$ ,  $b \in V^*$ . A preconditioner is then  $\mathcal{B} \in \mathcal{L}(V^*, V)$ , constructed such that  $\mathcal{B}$  is a positive, self-adjoint isomorphism. Then  $\mathcal{B}\mathcal{A} \in \mathcal{L}(V)$  is an isomorphism.*

*To discretize (2.8) we shall here employ finite element spaces  $V_h$  consisting of linear continuous finite elements where  $\Gamma_h$  is formed by the facets of  $\Omega_h$ , cf. Figure 3.1. Stability of discretizations of (2.7) (for the more general case where the discretization of  $\Omega$  and  $\Gamma$  are independent) is studied e.g. in [40] and [42, ch. 11.3].*

*The linear system resulting from discretization leads to the following system of equations*

$$\mathbb{B}\mathbf{A}\mathbf{x} = \mathbb{B}\mathbf{b}, \quad (2.9)$$

*where*

$$\mathbb{B} = \begin{bmatrix} \mathbf{A}^{-1} & \\ & \mathbf{H}(-\frac{1}{2})^{-1} \end{bmatrix} \quad \text{and} \quad \mathbf{A} = \begin{bmatrix} \mathbf{A} & \mathbf{B}^\top \\ \mathbf{B} & \end{bmatrix}.$$

*The last block of the matrix preconditioner  $\mathbb{B}$  is the inverse of the matrix constructed by (2.5) (using discretization of an operator inducing the  $H^1(\Gamma)$  norm on the second subspace of  $V_h$ ) and matrix  $\mathbb{B}\mathbf{A}$  has the same eigenvalues as operator  $\mathcal{B}_h\mathcal{A}_h$ .*

*Tables 2.1 and 2.2 consider the problem (2.7) with  $\Omega$  the unit square and  $\Gamma$  its left edge. In Table 2.1 we show the spectral condition number of the matrix  $\mathbb{B}\mathbf{A}$  as a function of the discretization parameter  $h$ . It is evident that the condition number is bounded by a constant.*

Table 2.1: The smallest and the largest eigenvalues and the spectral condition number of matrix  $\mathbb{B}\mathbb{A}$  from system (2.9).

$h$	$\lambda_{\min}$	$\lambda_{\max}$	$\kappa$
$1.77 \times 10^{-1}$	0.311	1.750	5.622
$8.84 \times 10^{-2}$	0.311	1.750	5.622
$4.42 \times 10^{-2}$	0.311	1.750	5.622
$2.21 \times 10^{-2}$	0.311	1.750	5.622
$1.11 \times 10^{-2}$	0.311	1.750	5.622

Table 2.2: The number of iterations required for convergence of the minimal residual method for system (2.9) with  $\mathbb{B}$  replaced by the approximation (2.10).

size	$n_{\text{iters}}$	$\ u - u_h\ _{1,\Omega}$
4290	38	$6.76 \times 10^{-2}(1.00)$
16770	40	$3.38 \times 10^{-2}(1.00)$
66306	38	$1.69 \times 10^{-2}(1.00)$
263682	38	$8.45 \times 10^{-3}(1.00)$
1051650	39	$4.23 \times 10^{-3}(1.00)$

Table 2.2 then reports the number of iterations required for convergence of the minimal residual method [38] with the system (2.9) of different sizes. The iterations are started from a random initial vector and for convergence it is required that  $\mathbf{r}_k$ , the  $k$ -th residuum, satisfies  $\mathbf{r}_k^\top \mathbb{B} \mathbf{r}_k < 10^{-10}$ . The operator  $\mathbb{B}$  is the spectrally equivalent approximation of  $\mathbb{B}$  given as<sup>1</sup>

$$\bar{\mathbb{B}} = \text{diag} \left( \text{AMG}(\mathbf{A}), \text{LU} \left( \mathbf{H} \left( -\frac{1}{2} \right) \right) \right). \quad (2.10)$$

The iteration count appears to be bounded independently of the size of the linear system.

Together the presented results indicate that the constructed preconditioner whose discrete approximation utilizes matrices (2.5) is a good preconditioner for system (2.6).

Finally, with  $\Omega \in \mathbb{R}^2$ ,  $\Gamma \subset \Omega$  of codimension one we consider the problem (1.1). The weak formulation of (1.1a)–(1.1c), using the method of Lagrange multipliers, defines a variational problem for the triplet  $(u, v, p) \in U \times V \times Q$

$$\begin{aligned} (\nabla u, \nabla \phi)_\Omega + \langle p, \epsilon T_\Gamma \phi \rangle_\Gamma &= (f, \phi)_\Omega & \phi &\in U, \\ (\nabla v, \nabla \psi)_\Gamma - \langle p, \psi \rangle_\Gamma &= (g, \psi)_\Gamma & \psi &\in V, \\ \langle \chi, \epsilon T_\Gamma u - v \rangle_\Gamma &= 0 & \chi &\in Q, \end{aligned} \quad (2.11)$$

where  $U, V, Q$  are Hilbert spaces to be specified later. The well-posedness of (2.11) is guaranteed provided that the celebrated Brezzi conditions, see Appendix A, are fulfilled. We remark that

$$\langle p, T_\Gamma \phi \rangle_\Gamma = \langle \delta_\Gamma p, \phi \rangle_\Omega.$$

Hence  $\delta_\Gamma$  is in our context the dual operator to the trace operator  $T_\Gamma$ . Since  $T_\Gamma : H_0^1(\Omega) \rightarrow H_{00}^{\frac{1}{2}}(\Gamma)$ , then  $\delta_\Gamma : H^{-\frac{1}{2}}(\Gamma) \rightarrow H^{-1}(\Omega)$ .

For our discussion of preconditioners it is suitable to recast (2.11) as an operator equation for the self-adjoint operator  $\mathcal{A}$

$$\mathcal{A} \begin{bmatrix} u \\ v \\ p \end{bmatrix} = \begin{bmatrix} A_U & B_U^* \\ A_V & B_V^* \\ B_U & B_V \end{bmatrix} \begin{bmatrix} u \\ v \\ p \end{bmatrix} = \begin{bmatrix} f \\ g \end{bmatrix} \quad (2.12)$$

<sup>1</sup> Here and in the subsequent numerical experiments AMG is the algebraic multigrid BOOMER-AMG from the Hypr library [23] and LU is the direct solver from the UMFPACK library [19]. The libraries were accessed through the interface provided by PETSc [7] version 3.5.3. To assemble the relevant matrices FEniCS library [31] version 1.6.0 and its extension for block-structured systems cbc.block [34] were used. The AMG preconditioner was used with the default options except for coarsening which was set to Ruge-Stueben algorithm.

with the operators  $A_i, B_i, i \in \{U, V\}$  given by

$$\begin{aligned}\langle A_U u, \phi \rangle_\Omega &= (\nabla u, \nabla \phi)_\Omega, & \langle A_V v, \psi \rangle_\Gamma &= (\nabla v, \nabla \psi)_\Gamma, \\ \langle B_U u, \chi \rangle_\Gamma &= \langle \chi, \epsilon T_\Gamma u \rangle_\Gamma, & \langle B_V v, \chi \rangle_\Gamma &= -\langle \chi, v \rangle_\Gamma.\end{aligned}$$

Further, for discussion of mapping properties of  $\mathcal{A}$  it will be advantageous to consider the operator as a map defined over space  $W \times Q$ ,  $W = U \times V$  as

$$\mathcal{A} = \begin{bmatrix} A & B^* \\ B & \end{bmatrix} \quad \text{with} \quad A = \begin{bmatrix} A_U & \\ & A_V \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} B_U & B_V \end{bmatrix}. \quad (2.13)$$

Considering two different choices of spaces  $U, V$  and  $Q$  we will propose two formulations that lead to different preconditioners

$$\mathcal{B}_Q^{-1} = \begin{bmatrix} A_U & & \\ & A_V & \\ & & B_U A_U^{-1} B_U^* + B_V A_V^{-1} B_V^* \end{bmatrix} \quad (2.14)$$

and

$$\mathcal{B}_W^{-1} = \begin{bmatrix} A_U + B_U^* R B_U & & \\ & A_V & \\ & & B_V A_V^{-1} B_V^* \end{bmatrix}. \quad (2.15)$$

Here  $R$  is the Riesz map from  $Q^*$  to  $Q$ . Preconditioners of the form (2.14)–(2.15) will be referred to as the  $Q$ -cap and the  $W$ -cap preconditioners. This naming convention reflects the role intersection spaces play in the respected formulations. We remark that the definitions should be understood as templates identifying the correct structure of the preconditioner.

**3.  $Q$ -cap preconditioner.** Consider operator  $\mathcal{A}$  from problem (2.12) as a mapping  $W \times Q \rightarrow W^* \times Q^*$ ,

$$\begin{aligned}W &= H_0^1(\Omega) \times H_0^1(\Gamma), \\ Q &= \epsilon H^{-\frac{1}{2}}(\Gamma) \cap H^{-1}(\Gamma).\end{aligned} \quad (3.1)$$

The spaces are equipped with norms

$$\|w\|_W^2 = |u|_{1,\Omega}^2 + |v|_{1,\Gamma}^2 \quad \text{and} \quad \|p\|_Q^2 = \epsilon^2 \|p\|_{-\frac{1}{2},\Gamma}^2 + \|p\|_{-1,\Gamma}^2. \quad (3.2)$$

Since  $H^{-\frac{1}{2}}(\Gamma)$  is continuously embedded in  $H^{-1}(\Gamma)$ , the space  $Q$  is the same topological vector space as  $H^{-\frac{1}{2}}(\Gamma)$ , but equipped with an equivalent,  $\epsilon$ -dependent inner product. See also [9, ch. 2]. The next theorem shows that this definition leads to a well-posed problem.

We will need a right inverse of the trace operator and employ the following harmonic extension. Let  $q \in H_{00}^{\frac{1}{2}}(\Gamma)$  and let  $u$  be the solution of the problem

$$\begin{aligned}-\Delta u &= 0, & \text{in } \Omega \setminus \Gamma, \\ u &= 0, & \text{on } \partial\Omega, \\ u &= q, & \text{on } \Gamma.\end{aligned} \quad (3.3)$$

Since trace is surjective onto  $H_{00}^{\frac{1}{2}}(\Gamma)$ , (3.3) has a solution  $u \in H_0^1(\Omega)$  and  $|u|_{1,\Omega} \leq C|q|_{\frac{1}{2},\Gamma}$  for some constant  $C$ . We denote the harmonic extension operator by  $E$ , i.e.,  $u = Eq$  with  $\|E\| \leq C$ .

**THEOREM 3.1.** *Let  $W$  and  $Q$  be the spaces (3.1). The operator  $\mathcal{A} : W \times Q \rightarrow W^* \times Q^*$ , defined in (2.12) is an isomorphism and the condition number of  $\mathcal{A}$  is bounded independently of  $\epsilon > 0$ .*

*Proof.* The statement follows from the Brezzi theorem A.1 once its assumptions are verified. Since  $A$  induces the inner product on  $W$ ,  $A$  is continuous and coercive and the conditions (A.1a) and (A.1b) hold. Next, we see that  $B$  is bounded,

$$\begin{aligned} \langle Bw, q \rangle_\Gamma &= \langle q, \epsilon T_\Gamma u - v \rangle_\Gamma \\ &\leq \|q\|_{-\frac{1}{2}, \Gamma} \|\epsilon T_\Gamma u\|_{\frac{1}{2}, \Gamma} + \|q\|_{-1, \Gamma} |v|_{1, \Gamma} \\ &\leq (1 + \|T_\Gamma\|) \sqrt{\epsilon^2 \|q\|_{-\frac{1}{2}, \Gamma}^2 + \|q\|_{-1, \Gamma}^2} \sqrt{|u|_{1, \Omega}^2 + |v|_{1, \Gamma}^2} \\ &= (1 + \|T_\Gamma\|) \|q\|_Q \|w\|_W. \end{aligned}$$

It remains to show the inf-sup condition (A.1d). Since the trace is bounded and surjective, for all  $\xi \in H_{00}^{\frac{1}{2}}(\Gamma)$  we let  $u$  be defined in terms of the harmonic extension (3.3) such that  $u = \epsilon^{-1} E\xi$  and  $|u|_{1, \Omega} \leq \epsilon^{-1} \|E\| \|\xi\|_{\frac{1}{2}, \Gamma}$ . Hence,

$$\begin{aligned} \sup_{w \in W} \frac{\langle Bw, q \rangle_\Gamma}{\|w\|_W} &= \sup_{w \in W} \frac{\langle q, \epsilon T_\Gamma u - v \rangle_\Gamma}{\sqrt{|u|_{1, \Omega}^2 + |v|_{1, \Gamma}^2}} \\ &\geq (1 + \|E\|)^{-1} \sup_{(\xi, v) \in H_{00}^{\frac{1}{2}}(\Gamma) \times H_0^1(\Gamma)} \frac{\langle q, \xi + v \rangle_\Gamma}{\sqrt{\epsilon^{-2} \|\xi\|_{\frac{1}{2}, \Gamma}^2 + \|v\|_{1, \Gamma}^2}} \end{aligned}$$

Note that we have the identity

$$Q^* = (\epsilon H^{-\frac{1}{2}}(\Gamma) \cap H^{-1}(\Gamma))^* = \epsilon^{-1} H_{00}^{\frac{1}{2}}(\Gamma) + H_0^1(\Gamma),$$

equipped with the norm

$$\|q^*\|_{Q^*} = \inf_{q^* = q_1^* + q_2^*} \epsilon^{-2} \|q_1^*\|_{\frac{1}{2}, \Gamma}^2 + |q_2^*|_{1, \Gamma}^2.$$

See also [9]. It follows that

$$\begin{aligned} \sup_{(\xi, v) \in H^{\frac{1}{2}}(\Gamma) \times H_0^1(\Gamma)} \frac{\langle q, \xi + v \rangle_\Gamma}{\sqrt{\epsilon^{-2} \|\xi\|_{\frac{1}{2}, \Gamma}^2 + \|v\|_{1, \Gamma}^2}} &= \sup_{\zeta \in Q^*} \sup_{\substack{\xi + v = \zeta \\ v \in H_0^1(\Gamma)}} \frac{\langle q, \xi + v \rangle_\Gamma}{\sqrt{\epsilon^{-2} \|\xi\|_{\frac{1}{2}, \Gamma}^2 + \|v\|_{1, \Gamma}^2}} \\ &= \sup_{\zeta \in Q^*} \frac{\langle q, \zeta \rangle_\Gamma}{\inf_{\substack{\xi + v = \zeta \\ v \in H_0^1(\Gamma)}} \sqrt{\epsilon^{-2} \|\xi\|_{\frac{1}{2}, \Gamma}^2 + \|v\|_{1, \Gamma}^2}} \\ &= \|q\|_{Q^{**}} = \|q\|_Q. \end{aligned}$$

Consequently, condition (A.1d) holds with a constant independent of  $\epsilon$ .  $\square$

Following Theorem 3.1 and [35] a preconditioner for the symmetric isomorphic operator  $\mathcal{A}$  is the Riesz mapping  $W^* \times Q^*$  to  $W \times Q$

$$\mathcal{B}_Q = \begin{bmatrix} -\Delta_\Omega & & \\ & -\Delta_\Gamma & \\ & & \epsilon^2 \Delta_\Gamma^{-\frac{1}{2}} + \Delta_\Gamma^{-1} \end{bmatrix}^{-1}. \quad (3.4)$$



Here  $\Delta_\Gamma^s$  is defined by  $\langle \Delta_\Gamma^s v, w \rangle_\Gamma = (v, w)_{H_s}$ , with the  $H_s$ -inner product defined by (2.4). Hence the norm induced on  $W \times Q$  by the operator  $\mathcal{B}_Q^{-1}$  is not (3.2) but an equivalent norm

$$\langle \mathcal{B}_Q^{-1} x, x \rangle = |u|_{1,\Omega}^2 + |v|_{1,\Gamma}^2 + \epsilon^2 \|p\|_{H_{-\frac{1}{2}}(\Gamma)}^2 + \|p\|_{H_{-1}(\Gamma)}^2$$

for any  $x = (u, v, p) \in W \times Q$ . Note that  $\mathcal{B}_Q$  fits the template defined in (2.14).

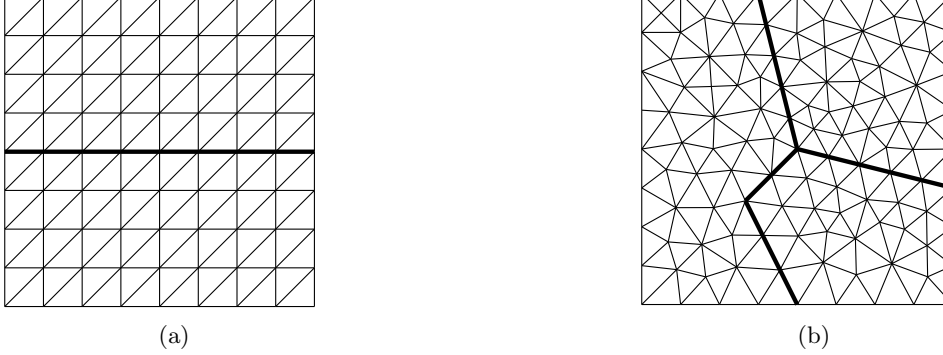


Fig. 3.1: Geometrical configurations and their sample triangulations considered in the numerical experiments.

**3.1. Discrete  $Q$ -cap preconditioner.** Following Theorem 3.1 the  $Q$ -cap preconditioner (3.4) is a good preconditioner for operator equation  $\mathcal{A}x = b$  with the condition number independent of the material parameter  $\epsilon$ . To translate the preconditioned operator equation  $\mathcal{B}_Q \mathcal{A}x = \mathcal{B}_Q b$  into a stable linear system it is necessary to employ suitable discretization. In particular, the Brezzi conditions must hold on each approximation space  $W_h \times Q_h$  with constants independent of the discretization parameter  $h$ . Such a suitable discretization will be referred to as stable.

Let us consider a stable discretization of operator  $\mathcal{A}$  from Theorem 3.1 by finite dimensional spaces  $U_h, V_h$  and  $Q_h$  defined as

$$U_h = \text{span} \{ \phi_i \}_{i=1}^{n_U}, \quad V_h = \text{span} \{ \psi_i \}_{i=1}^{n_V}, \quad Q_h = \text{span} \{ \chi_i \}_{i=1}^{n_Q}.$$

Then the Galerkin method for problem (2.12) reads: Find  $(u_h, v_h, p_h) \in U_h \times V_h \times Q_h$  such that

$$\begin{aligned} (\nabla u_h, \nabla \phi)_\Omega + \langle p_h, \epsilon T_\Gamma \phi \rangle_\Gamma &= (f, \phi)_\Omega & \phi &\in U_h, \\ (\nabla v_h, \nabla \psi)_\Gamma - \langle p_h, \psi \rangle_\Gamma &= (g, \psi)_\Gamma & \psi &\in V_h, \\ \langle \chi, \epsilon T_\Gamma u_h - v_h \rangle_\Gamma &= 0 & \chi &\in Q_h. \end{aligned}$$

Further we shall define matrices  $A_U, A_V$  and  $B_U, B_V$  in the following way

$$\begin{aligned} A_U &\in \mathbb{R}^{n_U \times n_U}, \quad (A_U)_{i,j} = (\nabla \phi_j, \nabla \phi_i)_\Omega, \\ A_V &\in \mathbb{R}^{n_V \times n_V}, \quad (A_V)_{i,j} = (\nabla \psi_j, \nabla \psi_i)_\Gamma, \\ B_U &\in \mathbb{R}^{n_Q \times n_U}, \quad (B_U)_{i,j} = \langle \epsilon T_\Gamma \phi_j, \chi_i \rangle_\Gamma, \\ B_V &\in \mathbb{R}^{n_Q \times n_V}, \quad (B_V)_{i,j} = -\langle \psi_j, \chi_i \rangle_\Gamma. \end{aligned} \tag{3.5}$$

We note that  $\mathbf{B}_V$  can be viewed as a representation of the negative identity mapping between spaces  $V_h$  and  $Q_h$ . Similarly, matrix  $\mathbf{B}_U$  can be viewed as a composite,  $\mathbf{B}_U = \mathbf{M}_{\bar{U}Q} \mathbf{T}$ . Here  $\mathbf{M}_{\bar{U}Q}$  is the representation of an identity map from space  $\bar{U}_h$  to space  $Q_h$ . The space  $\bar{U}_h$  is the image of  $U_h$  under the trace mapping  $T_\Gamma$ . We shall respectively denote the dimension of the space and its basis functions  $n_{\bar{U}}$  and  $\bar{\phi}_i$ ,  $i \in [1, n_{\bar{U}}]$ . Matrix  $\mathbf{T} \in \mathbb{R}^{n_{\bar{U}} \times n_U}$  is then a representation of the trace mapping  $T_\Gamma : U_h \rightarrow \bar{U}_h$ .

We note that the rank of  $\mathbf{T}$  is  $n_Q$  and mirroring the continuous operator  $T_\Gamma$  the matrix has a unique right inverse  $\mathbf{T}^+$ . We refer to [36] for the continuous case. The matrix  $\mathbf{T}^+$  can be computed as a pseudoinverse via the reduced singular value decomposition  $\mathbf{T}\mathbf{U} = \mathbf{Q}\Sigma$ , see e.g. [45, ch. 11]. Then  $\mathbf{T}^+ = \mathbf{U}\Sigma^{-1}\mathbf{Q}$ . Here, the columns of  $\mathbf{U}$  can be viewed as coordinates of functions  $\bar{\phi}_i$  zero-extended to  $\Omega$  such that they form the  $l^2$  orthonormal basis of the subspace of  $\mathbb{R}^{n_U}$  where the problem  $\mathbf{T}\mathbf{u} = \bar{\mathbf{u}}$  is solvable. Further the kernel of  $\mathbf{T}$  is spanned by  $n_U$ -vectors representing those functions in  $U_h$  whose trace on  $\Gamma$  is zero.

For the space  $U_h$  constructed by the finite element method with the triangulation of  $\Omega$  such that  $\Gamma$  is aligned with the element boundaries, cf. Figure 3.1, it is a consequence of the nodality of the basis that  $\mathbf{T}^+ = \mathbf{T}^\top$ .

With definitions (3.5) we use  $\mathbb{A}$  to represent the operator  $\mathcal{A}$  from (2.12) in the basis of  $W_h \times Q_h$

$$\mathbb{A} = \begin{bmatrix} \mathbf{A}_U & & \mathbf{B}_U^\top \\ & \mathbf{A}_V & \mathbf{B}_V^\top \\ \mathbf{B}_U & \mathbf{B}_V & \end{bmatrix}. \quad (3.6)$$

Finally a discrete  $Q$ -cap preconditioner is defined as a matrix representation of (3.4) with respect to the basis of  $W_h \times Q_h$

$$\mathbb{B}_Q = \begin{bmatrix} \mathbf{A}_U & & \\ & \mathbf{A}_V & \\ & & \epsilon^2 \mathbf{H}(-\frac{1}{2}) + \mathbf{H}(-1) \end{bmatrix}^{-1}. \quad (3.7)$$

The matrices  $\mathbf{A}$ ,  $\mathbf{M}$  which are used to compute the values  $\mathbf{H}(\cdot)$  through the definition (2.5) have the property  $|p|_{1,\Gamma}^2 = \mathbf{p}^\top \mathbf{A} \mathbf{p}$  and  $\|p\|_{0,\Gamma}^2 = \mathbf{p}^\top \mathbf{M} \mathbf{p}$  for every  $p \in Q_h$  and  $\mathbf{p} \in \mathbb{R}^{n_Q}$  its coordinate vector. Note that due to properties of matrices  $\mathbf{H}(\cdot)$ , matrix  $\mathbf{N}_Q$ , the inverse of the final block of  $\mathbb{B}_Q$ , is given by

$$\mathbf{N}_Q = [\epsilon^2 \mathbf{H}(-\frac{1}{2}) + \mathbf{H}(-1)]^{-1} = \mathbf{U} [\epsilon^2 \Lambda^{-\frac{1}{2}} + \Lambda^{-1}]^{-1} \mathbf{U}^\top. \quad (3.8)$$

By Theorem 3.1 and the assumption on spaces  $W_h \times Q_h$  being stable, the matrix  $\mathbb{B}_Q \mathbb{A}$  has a spectrum bounded independent of the parameter  $\epsilon$  and the size of the system or equivalently discretization parameter  $h$ . In turn  $\mathbb{B}_Q$  is a good preconditioner for matrix  $\mathbb{A}$ . To demonstrate this property we shall now construct a stable discretization of the space  $W \times Q$  using the finite element method.

**3.2. Stable subspaces for  $Q$ -cap preconditioner.** For  $h > 0$  fixed let  $\Omega_h$  be the polygonal approximation of  $\Omega$ . For the set  $\bar{\Omega}_h$  we construct a shape-regular triangulation consisting of closed triangles  $K_i$  such that  $\Gamma \cap K_i$  is an edge  $e_i$  of the triangle. Let  $\Gamma_h$  be a union of such edges. The discrete spaces  $W_h \subset W$  and  $Q_h \subset Q$  shall be defined in the following way. Let

$$\begin{aligned} U_h &= \{v \in C(\bar{\Omega}_h) : v|_K = \mathbb{P}_1(K)\}, \\ V_h &= \{v \in C(\bar{\Gamma}_h) : v|_e = \mathbb{P}_1(e)\}, \end{aligned} \quad (3.9)$$

where  $\mathbb{P}_1(D)$  are linear polynomials on the simplex  $D$ . Then we set

$$\begin{aligned} W_h &= (U_h \cap H_0^1(\Omega)) \times (V_h \cap H_0^1(\Gamma)), \\ Q_h &= V_h \cap H_0^1(\Gamma). \end{aligned} \quad (3.10)$$

Let  $A_h, B_h$  be the finite dimensional operators defined on the approximation spaces (3.10) in terms of Galerkin method for operators  $A, B$  in (2.13). Since the constructed spaces are conforming the operators  $A_h, B_h$  are continuous with respect to the norms (3.2). Further  $A_h$  is  $W$ -elliptic on  $W_h$  since the operator defines an inner product on the discrete space. Thus to show that the spaces  $W_h \times Q_h$  are stable it remains to show that the discrete inf-sup condition holds.

LEMMA 3.2. *Let  $W_h \subset W, Q_h \subset Q$  be the spaces (3.10). Further let  $\|\cdot\|_W, \|\cdot\|_Q$  be the norms (3.2). Finally let  $B_h$  such that  $\langle B_h w_h, q_h \rangle_\Gamma = \langle B w, q_h \rangle_\Gamma, w \in W$ . There exists a constant  $\beta > 0$  such that*

$$\inf_{q_h \in Q_h} \sup_{w_h \in W_h} \frac{\langle B_h w_h, q_h \rangle_\Gamma}{\|w_h\|_W \|q_h\|_Q} \geq \beta. \quad (3.11)$$

*Proof.* Recall  $Q = \epsilon H^{-\frac{1}{2}}(\Gamma) \cap H^{-1}(\Gamma)$ . We follow the steps of the continuous inf-sup condition in the reverse order. By definition

$$\begin{aligned} \|q_h\|_Q &= \sup_{p \in \epsilon H_{00}^{\frac{1}{2}}(\Gamma) + H_0^1(\Gamma)} \frac{\langle q_h, p \rangle_\Gamma}{\inf_{p=p_1+p_2} \sqrt{\epsilon^{-2} \|p_1\|_{\frac{1}{2},\Gamma}^2 + \|p_2\|_{1,\Gamma}^2}} \\ &= \sup_p \sup_{p=p_1+p_2} \frac{\langle q_h, p_1 \rangle_\Gamma + \langle q_h, p_2 \rangle_\Gamma}{\sqrt{\epsilon^{-2} \|p_1\|_{\frac{1}{2},\Gamma}^2 + \|p_2\|_{1,\Gamma}^2}}. \end{aligned} \quad (3.12)$$

For each  $p_1 \in H_{00}^{\frac{1}{2}}(\Gamma)$  let  $u_h \in U_h$  the weak solution of the boundary value problem

$$\begin{aligned} -\Delta u &= 0 && \text{in } \Omega, \\ \epsilon u &= p_1 && \text{on } \Gamma, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned}$$

Then  $\epsilon T_\Gamma u_h = p_1$  in  $H_{00}^{\frac{1}{2}}(\Gamma)$  and  $\epsilon |u_h|_{1,\Omega} \leq C \|p_1\|_{\frac{1}{2},\Gamma}$  for some constant  $C$  depending only on  $\Omega$  and  $\Gamma$ . For each  $p_2 \in H_0^1(\Gamma)$  let  $v_h \in V_h$  be the  $L^2$  projection of  $p_2$  onto the space  $V_h$

$$\langle v_h - p_2, z \rangle_\Gamma = 0 \quad z \in V_h. \quad (3.13)$$

By construction we then have  $\langle q_h, p_2 - v_h \rangle_\Gamma = 0$  for all  $q_h \in Q_h$  and

$\|v_h\|_{0,\Gamma} \leq \|p_2\|_{0,\Gamma}$ . Moreover for shape regular triangulation the projection  $\Pi : H_0^1(\Gamma) \rightarrow V_h, v_h = \Pi p_2$  is bounded in the  $H_0^1$  norm

$$|v_h|_{1,\Gamma} \leq |p_2|_{1,\Gamma}. \quad (3.14)$$

We refer to [10, ch. 7] for this result. For constructed  $u_h, v_h$  it follows from (3.12) that

$$\begin{aligned} \|q_h\|_Q &\lesssim \sup_{w_h \in U_h + V_h} \sup_{w_h = u_h + v_h} \frac{\langle q_h, \epsilon T_\Gamma u_h + v_h \rangle_\Gamma}{\sqrt{|u_h|_{1,\Omega}^2 + |v_h|_{1,\Gamma}^2}} \\ &= \sup_{(u_h, v_h) \in U_h \times V_h} \frac{\langle q_h, \epsilon T_\Gamma u_h + v_h \rangle_\Gamma}{\|(u_h, v_h)\|_W} = \sup_{w_h \in W_h} \frac{\langle B_h w_h, q_h \rangle_\Gamma}{\|w_h\|_W}. \end{aligned}$$

□ The constructed stable discretizations (3.10) are a special case of conforming spaces built from  $U_{h;k} \subset H^1(\Omega)$  and  $V_{h;l} \subset H^1(\Gamma)$  defined as

$$\begin{aligned} U_{h;k} &= \{v \in C(\bar{\Omega}_h) : v|_K = \mathbb{P}_k(K)\}, \\ V_{h;l} &= \{v \in C(\bar{\Gamma}_h) : v|_e = \mathbb{P}_l(e)\}. \end{aligned} \quad (3.15)$$

The following corollary gives a necessary compatibility condition on polynomial degrees in order to build inf-sup stable spaces from components (3.15).

**COROLLARY 3.3.** *Let  $W_{h;k,l} = (U_{h;k} \cap H_0^1(\Omega)) \times (V_{h;l} \cap H_0^1(\Gamma))$  and  $Q_{h;m} = V_{h;m} \cap H_0^1(\Gamma)$ . The necessary condition for (3.11) to hold with space  $W_{h;k,l} \times Q_{h;m}$  is that  $m \leq \max(k, l)$ .*

*Proof.* Note that  $T_\Gamma u_h - v_h$  is piecewise polynomial of degree  $\max(k, l)$ . Suppose  $m > \max(k, l)$ . Then for each  $(u_h, v_h) \in W_{h;k,l}$  we can find a orthogonal polynomial  $0 \neq q_h \in Q_{h;m}$  such that

$$\langle q_h, T_\Gamma u_h - v_h \rangle_\Gamma = 0.$$

In turn  $\beta = 0$  in (3.11) and the discrete inf-sup condition cannot hold. □

**3.3. Numerical experiments.** Let now  $\mathbb{A}, \mathbb{B}_Q$  be the matrices (3.6), (3.7) assembled over the constructed stable spaces (3.10). We demonstrate the robustness of the  $Q$ -cap preconditioner (3.4) through a pair of numerical experiments. First, the *exact* preconditioner represented by the matrix  $\mathbb{B}_Q$  is considered and we are interested in the condition number of  $\mathbb{B}_Q \mathbb{A}$  for different values of the parameter  $\epsilon$ . The spectral condition number is computed from the smallest and largest (in magnitude) eigenvalues of the generalized eigenvalue problem  $\mathbb{A}x = \lambda \mathbb{B}_Q^{-1}x$ , which is here solved by SLEPc<sup>2</sup> [27]. The obtained results are reported in Table 3.1. In general, the condition numbers are well-behaved indicating that  $\mathbb{B}_Q$  defines a parameter robust preconditioner. We note that for  $\epsilon \ll 1$  the spectral condition number is close to  $(1 + \sqrt{5})/(\sqrt{5} - 1) \approx 2.618$ . In §3.4 this observation is explained by the relation of the proposed preconditioner  $\mathbb{B}_Q$  and the matrix preconditioner of Murphy et al. [37].

Table 3.1: Spectral condition numbers of matrices  $\mathbb{B}_Q \mathbb{A}$  for the system assembled on geometry (a) in Figure 3.1.

size	$n_Q$	$\log_{10} \epsilon$						
		-3	-2	-1	0	1	2	3
99	9	2.655	2.969	4.786	6.979	7.328	7.357	7.360
323	17	2.698	3.323	5.966	7.597	7.697	7.715	7.717
1155	33	2.778	3.905	7.031	7.882	7.818	7.816	7.816
4355	65	2.932	4.769	7.830	8.016	7.855	7.843	7.843
16899	129	3.217	5.857	8.343	8.081	7.868	7.854	7.852
66563	257	3.710	6.964	8.637	8.113	7.872	7.856	7.855

In the second experiment, we monitor the number of iterations required for convergence of the MinRes method [38] (the implementation is provided by `cbc.block` [34]) applied to the preconditioned equation  $\mathbb{B}_Q \mathbb{A}x = \mathbb{B}_Q b$ . The operator  $\mathbb{B}_Q$  is an efficient

<sup>2</sup>We use generalized Davidson method with Cholesky preconditioner and convergence tolerance  $10^{-8}$ .

and spectrally equivalent approximation of  $\mathbb{B}_Q$ ,

$$\bar{\mathbb{B}}_Q = \begin{bmatrix} \text{AMG}(\mathbf{A}_U) & & \\ & \text{LU}(\mathbf{A}_V) & \\ & & \mathbf{N}_Q \end{bmatrix}, \quad (3.16)$$

with  $\mathbf{N}_Q$  defined in (3.8). The iterations are started from a random initial vector and as a stopping criterion a condition on the magnitude of the  $k$ -th preconditioned residual  $\mathbf{r}_k$ ,  $\mathbf{r}_k^\top \bar{\mathbb{B}}_Q \mathbf{r}_k < 10^{-12}$  is used. The observed number of iterations is shown in Table 3.2. Robustness with respect to size of the system and the material parameter is evident as the iteration count is bounded for all the considered discretizations and values of  $\epsilon$ .

Table 3.2: Iteration count for convergence of  $\bar{\mathbb{B}}_Q \mathbf{A} \mathbf{x} = \bar{\mathbb{B}}_Q \mathbf{b}$  solved with the minimal residual method. The problem is assembled on geometry (a) from Figure 3.1.

size	$n_Q$	$\log_{10} \epsilon$						
		-3	-2	-1	0	1	2	3
66563	257	20	34	37	32	28	24	21
264195	513	22	34	34	30	26	24	20
1052675	1025	24	33	32	28	26	22	18
4202499	2049	26	32	30	26	24	20	17
8398403	2897	26	30	30	26	22	19	15
11075583	3327	26	30	30	26	22	19	15

Comparing Tables 3.1 and 3.2 we observe that the  $\epsilon$ -behavior of the condition number and the iteration counts are different. In particular, fewer iterations are required for  $\epsilon = 10^3$  than for  $\epsilon = 10^{-3}$  while the condition number in the former case is larger. Moreover, the condition numbers for  $\epsilon > 1$  are almost identical whereas the iteration counts decrease as the parameter grows. We note that these observations should be viewed in the light of the fact that the convergence of the minimal residual method in general does not depend solely on the condition number, e.g. [29], and a more detailed knowledge of the eigenvalues is required to understand the behavior.

Having proved and numerically verified the properties of the  $Q$ -cap preconditioner, we shall in the next section link  $\mathbb{B}_Q$  to a block diagonal matrix preconditioner suggested by Murphy et al. [37]. Both matrices are assumed to be assembled on the spaces (3.10) and the main objective of the section is to prove spectral equivalence of the two preconditioners.

**3.4. Relation to Schur complement preconditioner.** Consider a linear system  $\mathbf{A} \mathbf{x} = \mathbf{b}$  with an indefinite matrix (3.6) which shall be preconditioned by a block diagonal matrix

$$\mathbb{B} = \text{diag}(\mathbf{A}_U, \mathbf{A}_V, \mathbf{S})^{-1}, \quad \mathbf{S} = \mathbf{B}_U \mathbf{A}_U^{-1} \mathbf{B}_U^\top + \mathbf{B}_V \mathbf{A}_V^{-1} \mathbf{B}_V^\top, \quad (3.17)$$

where  $\mathbf{S}$  is the negative Schur complement of  $\mathbf{A}$ . Following [37] the spectrum of  $\mathbb{B} \mathbf{A}$  consists of three distinct eigenvalues. In fact  $\rho(\mathbb{B} \mathbf{A}) = \{1, \frac{1}{2} \pm \frac{1}{2} \sqrt{5}\}$ . A suitable Krylov method is thus expected to converge in no more than three iterations. However in its presented form  $\mathbb{B}$  does not define an efficient preconditioner. In particular, the cost of setting up the Schur complement comes close to inverting the system matrix  $\mathbf{A}$ . Therefore a cheaply computable approximation of  $\mathbf{S}$  is needed to make the

preconditioner practical (see e.g. [8, ch. 10.1] for an overview of generic methods for constructing the approximation). We proceed to show that if spaces (3.10) are used for discretization, the Schur complement is more efficiently approximated with the inverse of the matrix  $\mathbf{N}_Q$  defined in (3.8).

Let  $W_h, Q_h$  be the spaces (3.10). Then the mass matrix  $\mathbf{M}_{\overline{U}Q} = \mathbf{M}_{VQ}$  (cf. discussion prior to (3.5)) and the matrix will be referred to as  $\mathbf{M}$ . Moreover let us set  $\mathbf{A}_V = \mathbf{A}$ . With these definitions the Schur complement of  $\mathbf{A}$  reads

$$\mathbf{S} = \epsilon^2 \mathbf{M} \mathbf{T} \mathbf{A}_U^{-1} \mathbf{T}^\top \mathbf{M} + \mathbf{M} \mathbf{A}^{-1} \mathbf{M}. \quad (3.18)$$

Further, note that such matrices  $\mathbf{A}, \mathbf{M}$  are suitable for constructing the approximation of the  $H_s$  norm on the space  $Q_h$  by the mapping (2.5). In particular,  $\mathbf{A}$  is such that  $|p|_{1,\Gamma}^2 = \mathbf{p}^\top \mathbf{A} \mathbf{p}$  with  $p \in Q_h$  and  $\mathbf{p} \in \mathbb{R}^{n_Q}$  its coordinate vector. In turn the inverse of the matrix  $\mathbf{N}_Q$  reads

$$\mathbf{N}_Q^{-1} = (\mathbf{M} \mathbf{U}) (\epsilon^2 \Lambda^{-\frac{1}{2}} + \Lambda^{-1}) (\mathbf{M} \mathbf{U})^\top = \epsilon^2 \mathbf{H}(-\frac{1}{2}) + \mathbf{H}(-1). \quad (3.19)$$

Recalling that  $\mathbf{H}(-1) = \mathbf{M} \mathbf{A}^{-1} \mathbf{M}$  and contrasting (3.18) with (3.19) the matrices differ only in the first terms. We shall first show that if the terms are spectrally equivalent then so are  $\mathbf{S}$  and  $\mathbf{N}_Q^{-1}$ .

**THEOREM 3.4.** *Let  $\mathbf{S}, \mathbf{N}_Q^{-1}$  be the matrices defined respectively in (3.18) and (3.19) and let  $n_Q$  be their size. Assume that there exist positive constants  $c_1, c_2$  dependent only on  $\Omega$  and  $\Gamma$  such that for every  $n_Q > 0$  and any  $\mathbf{p} \in \mathbb{R}^{n_Q}$*

$$c_1 \mathbf{p}^\top \mathbf{H}(-\frac{1}{2}) \mathbf{p} \leq \mathbf{p}^\top \mathbf{M} \mathbf{T} \mathbf{A}_U^{-1} \mathbf{T}^\top \mathbf{M} \mathbf{p} \leq c_2 \mathbf{p}^\top \mathbf{H}(-\frac{1}{2}) \mathbf{p}.$$

*Then for each  $n_Q > 0$  matrix  $\mathbf{S}$  is spectrally equivalent with  $\mathbf{N}_Q^{-1}$ .*

*Proof.* By direct calculation we have

$$\begin{aligned} \mathbf{p}^\top \mathbf{S} \mathbf{p} &= \epsilon^2 \mathbf{p}^\top \mathbf{M} \mathbf{T} \mathbf{A}_U^{-1} \mathbf{T}^\top \mathbf{M} \mathbf{p} + \mathbf{p}^\top \mathbf{H}(-1) \mathbf{p} \\ &\leq c_2 \epsilon^2 \mathbf{p}^\top \mathbf{H}(-\frac{1}{2}) \mathbf{p} + \mathbf{p}^\top \mathbf{H}(-1) \mathbf{p} \\ &\leq C_2 \mathbf{p}^\top \mathbf{N}_Q^{-1} \mathbf{p} \end{aligned}$$

for  $C_2 = \sqrt{1 + c_2^2}$ . The existence of lower bound follows from estimate

$$\mathbf{p}^\top \mathbf{S} \mathbf{p} \geq c_1 \epsilon^2 \mathbf{p}^\top \mathbf{H}(-\frac{1}{2}) \mathbf{p} + \mathbf{p}^\top \mathbf{H}(-1) \mathbf{p} \geq C_1 \mathbf{p}^\top \mathbf{N}_Q^{-1} \mathbf{p}$$

with  $C_1 = \min(1, c_1)$ .  $\square$

Spectral equivalence of preconditioners  $\mathbb{B}_Q$  and  $\mathbb{B}$  now follows immediately from Theorem 3.4. Note that for  $\epsilon \ll 1$  the term  $\mathbf{H}(-1)$  dominates both  $\mathbf{S}$  and  $\mathbf{N}_Q^{-1}$ . In turn, the spectrum of  $\mathbb{B}\mathbf{A}$  is expected to approximate well the eigenvalues of  $\mathbb{B}_Q\mathbf{A}$ . This is then a qualitative explanation of why the spectral condition numbers of  $\mathbb{B}_Q\mathbf{A}$  observed for  $\epsilon = 10^{-3}$  in Table 3.1 are close to  $(1 + \sqrt{5})/(\sqrt{5} - 1)$ . It remains to prove that the assumption of Theorem 3.4 holds.

**LEMMA 3.5.** *There exist constants  $c_1, c_2 > 0$  depending only on  $\Omega, \Gamma$  such that for all  $n_Q > 0$  and  $p \in \mathbb{R}^{n_Q}$*

$$c_1 \mathbf{p}^\top \mathbf{H}(-\frac{1}{2}) \mathbf{p} \leq \mathbf{p}^\top \mathbf{M} \mathbf{T} \mathbf{A}_U^{-1} \mathbf{T}^\top \mathbf{M} \mathbf{p} \leq c_2 \mathbf{p}^\top \mathbf{H}(-\frac{1}{2}) \mathbf{p}.$$

*Proof.* For the sake of readability let  $n = n_Q$  and  $m = n_U$ . Since  $\mathbf{M}$  is symmetric and invertible,  $\mathbf{H}(-\frac{1}{2}) = \mathbf{M}\mathbf{U}\mathbf{A}^{-\frac{1}{2}}\mathbf{U}^\top\mathbf{M}$  and  $\mathbf{U}\mathbf{A}^{-\frac{1}{2}}\mathbf{U}^\top = \mathbf{H}(\frac{1}{2})^{-1}$  the statement is equivalent to

$$c_1 \mathbf{y}^\top \mathbf{H}(\frac{1}{2})^{-1} \mathbf{y} \leq \mathbf{y}^\top \mathbf{T} \mathbf{A}_U^{-1} \mathbf{T}^\top \mathbf{y} \leq c_2 \mathbf{y}^\top \mathbf{H}(\frac{1}{2})^{-1} \mathbf{y} \quad \text{for all } \mathbf{y} \in \mathbb{R}^m. \quad (3.20)$$

The proof is based on properties of the continuous trace operator  $T_\Gamma$ . Recall the trace inequality: There exists a positive constant  $K_2 = K_2(\Omega, \Gamma)$  such that  $\|T_\Gamma u\|_{\frac{1}{2}, \Gamma} \leq K_2 |u|_{1, \Omega}$  for all  $u \in H_0^1(\Omega)$ . From here it follows that the sequence  $\{\lambda_m^{\max}\}$ , where for each  $m$  value  $\lambda_m^{\max}$  is the largest eigenvalue of the eigenvalue problem

$$\mathbf{T}^\top \mathbf{H}(\frac{1}{2}) \mathbf{T} \mathbf{u} = \lambda \mathbf{A}_U \mathbf{u}, \quad (3.21)$$

is bounded from above by  $K_2$ . Note that the eigenvalue problem can be solved with nontrivial eigenvalue only for  $\mathbf{u} \in \mathbb{R}^n$  for which there exists some  $\mathbf{q} \in \mathbb{R}^m$  such that  $\mathbf{u} = \mathbf{T}^\top \mathbf{q}$ . Consequently the eigenvalue problem becomes  $\mathbf{T}^\top \mathbf{H}(\frac{1}{2}) \mathbf{q} = \lambda \mathbf{A}_U \mathbf{T}^\top \mathbf{q}$ . Next, applying the inverse of  $\mathbf{A}_U$  and the trace matrix yields  $\mathbf{T} \mathbf{A}_U^{-1} \mathbf{T}^\top \mathbf{H}(\frac{1}{2}) \mathbf{q} = \lambda \mathbf{q}$ . Finally, setting  $\mathbf{q} = \mathbf{H}(\frac{1}{2})^{-1} \mathbf{p}$  yields

$$\mathbf{T} \mathbf{A}_U^{-1} \mathbf{T}^\top \mathbf{p} = \lambda \mathbf{H}(\frac{1}{2})^{-1} \mathbf{p}. \quad (3.22)$$

Thus the largest eigenvalues of (3.21) and (3.22) coincide and in turn  $C_2 = K_2$ . Further (3.22) has only positive eigenvalues and the smallest nonzero eigenvalue of (3.21) is the smallest eigenvalue  $\lambda_m^{\min}$  of (3.22). Therefore for all  $\mathbf{y} \in \mathbb{R}^m$  it holds that  $\lambda_m^{\min} \mathbf{y}^\top \mathbf{H}(\frac{1}{2})^{-1} \mathbf{y} \leq \mathbf{y}^\top \mathbf{T} \mathbf{A}_U^{-1} \mathbf{T}^\top \mathbf{y}$ . But the sequence  $\{\lambda_m^{\min}\}$  is bounded from below since the right-inverse of the trace operator is bounded [36].  $\square$

The proof of Lemma 3.5 suggests that the constants  $c_1, c_2$  for spectral equivalence are computable as the limit of convergent sequences  $\{\lambda_m^{\min}\}, \{\lambda_m^{\max}\}$  consisting of the smallest and largest eigenvalues of the generalized eigenvalue problem (3.22). Convergence of such sequences for the two geometries in Figure 3.1 is shown in Figure 3.2. For the simple geometry (a) the sequences converge rather fast and the equivalence constants  $c_1, c_2$  are clearly visible in the figure. Convergence on the more complex geometry (b) is slower.

So far we have by Theorem 3.1 and Lemma 3.2 that the condition numbers of matrices  $\mathbb{B}_Q \mathbb{A}$  assembled over spaces (3.10) are bounded by constants independent of  $\{h, \epsilon\}$ . A more detailed characterization of the spectrum of the system preconditioned by the  $Q$ -cap preconditioner is given next. In particular, we relate the spectrum to computable bounds  $C_1, C_2$  and characterize the distribution of eigenvalues. Further, the effect of varying  $\epsilon$  (cf. Tables 3.1–3.2) is illustrated by numerical experiment.

**3.5. Spectrum of the  $Q$ -cap preconditioned system.** In the following, the left-right preconditioning of  $\mathbb{A}$  based on  $\mathbb{B}_Q$  is considered and we are interested in the spectrum of

$$\mathbb{B}_Q^\frac{1}{2} \mathbb{A} \mathbb{B}_Q^\frac{1}{2} = \begin{bmatrix} \mathbf{I}_U & & \mathbf{A}_U^{-\frac{1}{2}} \mathbf{B}_U^\top \mathbf{N}_Q^\frac{1}{2} \\ & \mathbf{I}_V & \mathbf{A}_V^{-\frac{1}{2}} \mathbf{B}_V^\top \mathbf{N}_Q^\frac{1}{2} \\ \mathbf{N}_Q^\frac{1}{2} \mathbf{B}_U \mathbf{A}_U^{-\frac{1}{2}} & \mathbf{N}_Q^\frac{1}{2} \mathbf{B}_V \mathbf{A}_V^{-\frac{1}{2}} & \end{bmatrix}. \quad (3.23)$$

The spectra of the left preconditioner system  $\mathbb{B}_Q \mathbb{A}$  and the left-right preconditioned system  $\mathbb{B}_Q^\frac{1}{2} \mathbb{A} \mathbb{B}_Q^\frac{1}{2}$  are identical. Using results of [41] the spectrum  $\rho$  of (3.23) is such

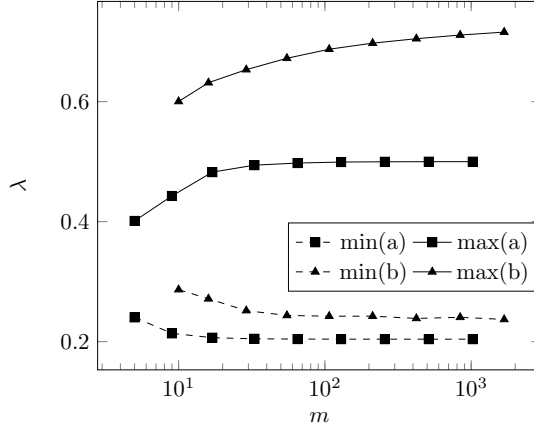


Fig. 3.2: Convergence of sequences  $\{\lambda_m^{\max}\}$   $\{\lambda_m^{\min}\}$  from Lemma 3.5 for geometries in Figure 3.1. For all sequences but  $\max(b)$  the constant bound is reached within the considered range of discretization parameter  $m = n_Q$ .

that  $\rho = I^- \cup I^+$  with

$$I^- = \left[ \frac{1 - \sqrt{1 + 4\sigma_{\max}^2}}{2}, \frac{1 - \sqrt{1 + 4\sigma_{\min}^2}}{2} \right] \quad I^+ = \left[ 1, \frac{1 + \sqrt{1 + 4\sigma_{\max}^2}}{2} \right] \quad (3.24)$$

and  $\sigma_{\min}, \sigma_{\max}$  the smallest and largest singular values of the block matrix formed by the first two row blocks in the last column of  $\mathbb{B}_Q^{\frac{1}{2}} \mathbb{A} \mathbb{B}_Q^{\frac{1}{2}}$ . We shall denote the matrix as  $\mathbb{D}$ ,

$$\mathbb{D} = \begin{bmatrix} \mathbf{A}_U^{-\frac{1}{2}} \mathbf{B}_U^{\top} \mathbf{N}_Q^{\frac{1}{2}} \\ \mathbf{A}_V^{-\frac{1}{2}} \mathbf{B}_V^{\top} \mathbf{N}_Q^{\frac{1}{2}} \end{bmatrix}.$$

PROPOSITION 3.6. *The condition number  $\kappa(\mathbb{B}_Q \mathbb{A})$  is bounded such that*

$$\kappa(\mathbb{B}_Q \mathbb{A}) \leq \frac{1 + \sqrt{1 + 4C_2}}{1 - \sqrt{1 + 4C_1}},$$

where  $C_1, C_2$  are the spectral equivalence bounds from Theorem 3.4.

*Proof.* Note that the singular values of matrix  $\mathbb{D}$  and the eigenvalues of matrix  $\mathbf{N}_Q^{\frac{1}{2}} \mathbf{S} \mathbf{N}_Q^{\frac{1}{2}}$  are identical. Further, using Theorem 3.4 with  $\mathbf{p} = \mathbf{N}_Q^{\frac{1}{2}} \mathbf{q}$ ,  $\mathbf{q} \in \mathbb{R}^{n_Q}$  yields

$$C_1 \mathbf{q}^{\top} \mathbf{q} \leq \mathbf{q}^{\top} \mathbf{N}_Q^{\frac{1}{2}} \mathbf{S} \mathbf{N}_Q^{\frac{1}{2}} \mathbf{q} \leq C_2 \mathbf{q}^{\top} \mathbf{q} \quad \text{for all } \mathbf{q} \in \mathbb{R}^{n_Q}.$$

In turn the spectrum of matrices  $\mathbf{N}_Q^{\frac{1}{2}} \mathbf{S} \mathbf{N}_Q^{\frac{1}{2}}$  is contained in the interval  $[C_1, C_2]$ . The statement now follows from (3.24).  $\square$  From numerical experiments we observe that the bound due to Proposition 3.6 slightly overestimates the condition number of the system. For example, using numerical trace bounds (cf. Figure 3.2) of geometry (a) in Figure 3.1,  $c_1 = 0.204$ ,  $c_2 = 0.499$  and Theorem 3.4, the formula yields 9.607 as the upper bound on the condition number. On the other hand condition numbers reported in Table 3.1 do not exceed 8.637. Similarly using estimated bounds for geometry (b)



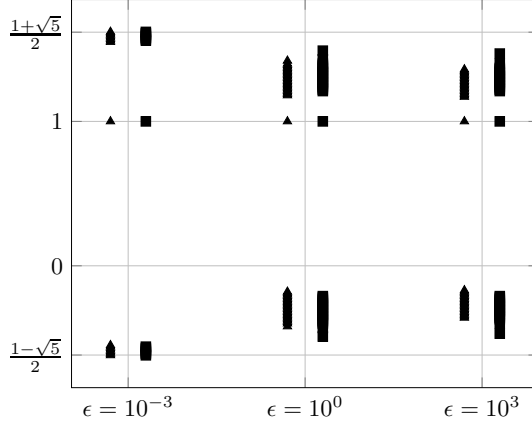


Fig. 3.3: Eigenvalues of matrices  $\mathbb{B}_Q \mathbb{A}$  assembled on geometries from Figure 3.1 for three different values of  $\epsilon$ . The value of  $\epsilon$  is indicated by grey vertical lines. On the left side of the lines is the spectrum for configuration (a). The spectrum for geometry (b) is then plotted on the right side. For  $\epsilon \ll 1$  the eigenvalues cluster near  $\lambda = 1$  and  $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$  (indicated by grey horizontal lines) which form the spectrum of  $\mathbb{B}\mathbb{A}$ .

$c_1 = 0.237, c_2 = 0.716$  the formula gives upper bound 8.676. The largest condition number in our experiments (not reported here) was 7.404.

It is clear that (3.24) could be used to analyze the effect of the parameter  $\epsilon$  on the spectrum provided that the singular values  $\sigma_{\min}, \sigma_{\max}$  were given as functions of  $\epsilon$ . We do not attempt to give this characterization here. Instead the effect of  $\epsilon$  is illustrated by a numerical experiment. Figure 3.3 considers the spectrum of  $\mathbb{B}_Q \mathbb{A}$  assembled on geometries from Figure 3.1 and three different values of the parameter. The systems from the two geometrical configurations are similar in size, 4355 for (a) and 4493 for (b). Note that for  $\epsilon \ll 1$  the eigenvalues for both configurations cluster near  $\lambda = 1$  and  $\lambda = \frac{1}{2} \pm \frac{1}{2}\sqrt{5}$ , that is, near the eigenvalues of  $\mathbb{B}\mathbb{A}$ . This observation is expected in the light of the discussion following Theorem 3.4. With  $\epsilon$  increasing the difference between  $\mathbb{B}_Q$  and  $\mathbb{B}$  caused by  $\mathbf{H}(-\frac{1}{2})$  becomes visible as the eigenvalues are no longer clustered. Observe that in these cases the lengths of intervals  $I^-, I^+$  are greater for geometry (b). This observation can be qualitatively understood via Proposition 3.6, Theorem 3.4 and Figure 3.2 where the trace map constants  $c_1, c_2$  of configuration (a) are more spread than those of (b).

**4.  $W$ -cap preconditioner.** To circumvent the need for mappings involving fractional Sobolev spaces we shall next study a different preconditioner for (2.11). As will be seen the new preconditioner  $W$ -cap preconditioner (2.15) is still robust with respect to the material and discretization parameters.

Consider operator  $\mathcal{A}$  from problem (2.12) as a mapping  $W \times Q \rightarrow W^* \times Q^*$ , with spaces  $W, Q$  defined as

$$\begin{aligned} W &= (H_0^1(\Omega) \cap \epsilon H_0^1(\Gamma)) \times H_0^1(\Gamma), \\ Q &= H^{-1}(\Gamma). \end{aligned} \quad (4.1)$$

The spaces are equipped with norms

$$\|w\|_W^2 = |u|_{1,\Omega}^2 + \epsilon^2 |T_\Gamma u|_{1,\Gamma}^2 + |v|_{1,\Gamma}^2 \quad \text{and} \quad \|p\|_Q^2 = \|p\|_{-1,\Gamma}^2. \quad (4.2)$$

Note that the trace of functions from space  $U$  is here controlled in the norm  $|\cdot|_{1,\Gamma}$  and not the fractional norm  $\|\cdot\|_{\frac{1}{2},\Gamma}$  as was the case in §3. Also note that the space  $W$  now is dependent on  $\epsilon$  while  $Q$  is not. The following result establishes well posedness of (2.11) with the above spaces.

**THEOREM 4.1.** *Let  $W$  and  $Q$  be the spaces (4.1). The operator  $\mathcal{A} : W \times Q \rightarrow W^* \times Q^*$ , defined in (2.12) is an isomorphism and the condition number of  $\mathcal{A}$  is bounded independently of  $\epsilon > 0$ .*

*Proof.* The proof proceeds by verifying the Brezzi conditions A.1. With  $w = (u, v)$ ,  $\omega = (\phi, \psi)$  application of the Cauchy-Schwarz inequality yields

$$\begin{aligned} \langle Aw, \omega \rangle_\Omega &= (\nabla u, \nabla \phi)_\Omega + (\nabla v, \nabla \psi)_\Gamma \\ &\leq |u|_{1,\Omega} |\phi|_{1,\Omega} + |v|_{1,\Gamma} |\psi|_{1,\Gamma} \\ &\leq |u|_{1,\Omega} |\phi|_{1,\Omega} + \epsilon^2 |T_\Gamma u|_{1,\Gamma} |\phi|_{1,\Gamma} + |v|_{1,\Gamma} |\psi|_{1,\Gamma} \\ &\leq \|w\|_W \|\omega\|_W. \end{aligned}$$

Therefore  $A$  is bounded with  $\|A\| = 1$  and (A.1a) holds. The coercivity of  $A$  on  $\ker B$  for (A.1b) is obtained from

$$\begin{aligned} \inf_{w \in \ker B} \frac{\langle Aw, w \rangle_\Omega}{\|w\|_W^2} &= \inf_{w \in \ker B} \frac{|u|_{1,\Omega}^2 + |v|_{1,\Gamma}^2}{|u|_{1,\Omega}^2 + \epsilon^2 |T_\Gamma u|_{1,\Gamma}^2 + |v|_{1,\Gamma}^2} \\ &= \inf_{w \in \ker B} \frac{|u|_{1,\Omega}^2 + |v|_{1,\Gamma}^2}{|u|_{1,\Omega}^2 + 2|v|_{1,\Gamma}^2} \geq \frac{1}{2}, \end{aligned}$$

where we used that  $\epsilon T_\Gamma u = v$  a.e. on the kernel. Consequently  $\alpha = \frac{1}{2}$ . Boundedness of  $B$  in (A.1c) with a constant  $\|B\| = \sqrt{2}$  follows from the Cauchy-Schwarz inequality

$$\begin{aligned} \langle Bw, q \rangle_\Gamma &\leq \|q\|_{-1,\Gamma} \epsilon |T_\Gamma u|_{1,\Gamma} + \|q\|_{-1,\Gamma} |v|_{1,\Gamma} \\ &\leq \sqrt{2} \|q\|_Q \sqrt{\epsilon^2 |T_\Gamma u|_{1,\Gamma}^2 + |v|_{1,\Gamma}^2} \\ &\leq \sqrt{2} \|q\|_Q \sqrt{|u|_{1,\Omega}^2 + \epsilon^2 |T_\Gamma u|_{1,\Gamma}^2 + |v|_{1,\Gamma}^2} \\ &\leq \sqrt{2} \|q\|_Q \|w\|_W. \end{aligned}$$

To show that the inf-sup condition holds compute

$$\begin{aligned} \sup_{w \in W} \frac{\langle Bw, q \rangle_\Gamma}{\|w\|_W} &= \sup_{w \in W} \frac{\langle q, \epsilon T_\Gamma u - v \rangle_\Gamma}{\sqrt{|u|_{1,\Omega}^2 + \epsilon^2 |T_\Gamma u|_{1,\Gamma}^2 + |v|_{1,\Gamma}^2}} \\ &\stackrel{u=0}{\geq} \sup_{v \in V} \frac{\langle q, v \rangle_\Gamma}{|v|_{1,\Gamma}} = \|q\|_Q. \end{aligned}$$

Thus  $\beta = 1$  in condition (A.1d).  $\square$

Following Theorem 4.1 the operator  $\mathcal{A}$  is a symmetric isomorphism between spaces  $W \times Q$  and  $W^* \times Q^*$ . As a preconditioner we shall consider a symmetric positive-definite isomorphism  $W^* \times Q^* \rightarrow W \times Q$

$$\mathcal{B}_W = \begin{bmatrix} (-\Delta_\Omega + T_\Gamma^* (-\epsilon^2 \Delta_\Gamma) T_\Gamma)^{-1} & & \\ & (-\Delta_\Gamma)^{-1} & \\ & & -\Delta_\Gamma \end{bmatrix}. \quad (4.3)$$

**4.1. Discrete preconditioner.** Similar to §3.1 we shall construct discretizations  $W_h \times Q_h$  of space  $W \times Q$  (4.1) such that the finite dimensional operator  $\mathcal{A}_h$  defined by considering  $\mathcal{A}$  from (2.12) on the constructed spaces satisfies the Brezzi conditions A.1.

Let  $W_h \subset W$  and  $Q_h \subset Q$  the spaces (3.10) of continuous piecewise linear polynomials. Then  $A_h, B_h$  are continuous with respect to norms (4.2) and it remains to verify conditions (A.1a) and (A.1d). First, coercivity of  $A_h$  is considered.

LEMMA 4.2. *Let  $W_h, Q_h$  the spaces (3.10) and  $A_h, B_h$  such that  $\langle Aw, \omega_h \rangle_\Omega = \langle A_h w_h, \omega_h \rangle_\Omega$ ,  $\langle Bw, q_h \rangle_\Gamma = \langle B_h w_h, q_h \rangle_\Gamma$ , for  $\omega_h, w_h \in W_h$ ,  $w \in W$  and  $q_h \in Q_h$ . Then there exists a constant  $\alpha > 0$  such that for all  $z_h \in \ker B_h$*

$$\langle A_h z_h, z_h \rangle \geq \alpha \|z_h\|_W,$$

where  $\|\cdot\|_W$  is defined in (4.2).

*Proof.* The claim follows from coercivity of  $A$  over  $\ker B$  (cf. Theorem 4.1) and the property  $\ker B_h \subset \ker B$ . To see that the inclusion holds, let  $z_h \in \ker B_h$ . Since  $z_h$  is continuous on  $\Gamma$  we have from definition  $\langle z_h, q_h \rangle_\Gamma = 0$  for all  $q_h \in Q_h$  that  $z_h|_\Gamma = 0$ . But then  $\langle z_h, q \rangle = 0$  for all  $q \in Q$  and therefore  $z_h \in \ker B$ .  $\square$

Finally, to show that the discretization  $W_h \times Q_h$  is stable we show that the inf-sup condition for  $B_h$  holds.

LEMMA 4.3. *Let spaces  $W_h, Q_h$  and operator  $B_h$  from Lemma 4.2. Then there exists  $\beta > 0$  such that*

$$\inf_{q_h \in Q_h} \sup_{w_h \in W_h} \frac{\langle B_h w_h, q_h \rangle_\Gamma}{\|w_h\|_W \|q_h\|_Q} \geq \beta, \quad (4.4)$$

where  $\|\cdot\|_Q$  is defined in (4.2).

*Proof.* We first proceed as in the proof of Theorem 4.1 and compute

$$\sup_{w_h \in W_h} \frac{\langle q_h, \epsilon T_\Gamma u_h - v_h \rangle_\Gamma}{\|w_h\|_W} \stackrel{u_h=0}{\geq} \sup_{v_h \in V_h} \frac{\langle v_h, q_h \rangle_\Gamma}{|v_h|_{1,\Gamma}}. \quad (4.5)$$

Next, for each  $p \in H_0^1(\Gamma)$  let  $v_h = \Pi p$  the element of  $V_h$  defined in the proof of Lemma 3.2. In particular, it holds that

$$\langle p - v_h, q_h \rangle_\Gamma = 0, \quad q_h \in Q_h$$

and  $|v_h|_{1,\Gamma} \leq C|p|_{1,\Gamma}$  for some constant  $C$  depending only on  $\Omega$  and  $\Gamma$ . Then

$$\|q_h\|_{-1,\Gamma} = \sup_{p \in H_0^1(\Gamma)} \frac{\langle q_h, p \rangle_\Gamma}{|p|_{1,\Gamma}} \leq C \sup_{v_h \in V_h} \frac{\langle q_h, v_h \rangle_\Gamma}{|v_h|_{1,\Gamma}}.$$

The estimate together with (4.5) proves the claim of the lemma.  $\square$

Let now  $\mathbf{A}_U, \mathbf{A}_V$  and  $\mathbf{B}_U, \mathbf{B}_V$  the matrices defined in (3.5) as representations of the corresponding finite dimensional operators in the basis of the stable spaces  $W_h$  and  $Q_h$ . We shall represent the preconditioner  $\mathcal{B}_W$  by a matrix

$$\mathbb{B}_W = \begin{bmatrix} (\mathbf{A}_U + \epsilon^2 \mathbf{T}^\top \mathbf{A} \mathbf{T})^{-1} & & \\ & (\mathbf{A}_V)^{-1} & \\ & & \mathbf{H}(-1)^{-1} \end{bmatrix}, \quad (4.6)$$

where  $H(-1)^{-1} = M^{-1}AM^{-1}$ , cf. (2.5), and  $M, A$  the matrices inducing  $L^2$  and  $H_0^1$  inner products on  $Q_h$ . Let us point out that there is an obvious correspondence between the matrix preconditioner  $\mathbb{B}_W$  and the operator  $\mathcal{B}_W$  defined in (2.15). On the other hand it is not entirely straight forward that the matrix  $\mathbb{B}_W$  represents the  $W$ -cap preconditioner defined here in (4.3). In particular, since the isomorphism from  $Q^* = H_0^1(\Gamma)$  to  $Q = H^{-1}(\Gamma)$  is realized by the Laplacian a case could be made for using the stiffness matrix  $A$  as a suitable representation of the operator.

Let us first argue for  $A$  not being a suitable representation for preconditioning. Note that the role of matrix  $A \in \mathbb{R}^{m \times n}$  in a linear system  $Ax = b$  is to transform vectors from the solution space  $\mathbb{R}^n$  to the residual space  $\mathbb{R}^m$ . In case the matrix is invertible the spaces coincide. However, to emphasize the conceptual difference between the spaces, let us write  $A : \mathbb{R}^n \rightarrow \mathbb{R}^{n*}$ . Then a preconditioner matrix is a mapping  $\mathbb{B} : \mathbb{R}^{n*} \rightarrow \mathbb{R}^n$ . The stiffness matrix  $A$ , however, is such that  $A : \mathbb{R}^{n_Q} \rightarrow \mathbb{R}^{n_Q*}$ .

It remains to show that  $M^{-1}AM^{-1}$  is the correct representation of  $A = -\Delta_\Gamma$ . Recall that  $Q_h \subset Q^*$  and  $A$  is the matrix representation of operator  $A_h : Q_h \rightarrow Q_h^*$ . Further, mappings  $\pi_h : Q_h \rightarrow \mathbb{R}^{n_Q}$ ,  $\mu_h : Q_h^* \rightarrow \mathbb{R}^{n_Q*}$

$$p_h = \sum_j (\pi_h p_h)_j \chi_j, \quad p_h \in Q_h \quad \text{and} \quad (\mu_h f_h)_j = \langle f_j, \chi_j \rangle, \quad f_h \in Q_h^*$$

define isomorphisms between<sup>3</sup>spaces  $Q_h, \mathbb{R}^{n_Q}$  and  $Q_h^*, \mathbb{R}^{n_Q*}$  respectively. We can uniquely associate each  $p_h \in Q_h$  with a functional in  $Q_h^*$  via the Riesz map  $I_h : Q_h \rightarrow Q_h^*$  defined as  $\langle I_h p_h, q_h \rangle_\Gamma = (p_h, q_h)_\Gamma$ . Since

$$(\mu_h I_h p_h)_j = (I_h p_h, \chi_j)_\Gamma = \sum_i (\pi_h p_h)_i (\chi_i, \chi_j)_\Gamma$$

the operator  $I_h$  is represented as the mass matrix  $M$ . The matrix then provides a natural isomorphism from  $\mathbb{R}^{n_Q}$  to  $\mathbb{R}^{n_Q*}$ . In turn  $M^{-1}AM^{-1} : \mathbb{R}^{n_Q*} \rightarrow \mathbb{R}^{n_Q}$  has the desired mapping properties. In conclusion, the inverse of the mass matrix was used in (4.6) as a natural adapter to obtain a matrix operating between spaces suitable for preconditioning.

Finally, we make a few observations about the matrix preconditioner  $\mathbb{B}_W$ . Recall that the  $Q$ -cap preconditioner  $\mathbb{B}_Q$  could be related to the Schur complement based preconditioner (3.17) obtained by factorizing  $A$  in (3.6). The relation of  $A$  to the  $W$ -cap preconditioner matrix (4.6) is revealed in the following calculation

$$\text{ULA} = \begin{bmatrix} A_V + \epsilon^2 T^\top A T & \tau^2 A & -M \\ -\epsilon M T & & M A^{-1} M \end{bmatrix}, \quad (4.7)$$

where

$$\mathbb{U} = \begin{bmatrix} I & & -T^\top \epsilon A M^{-1} \\ & I & \\ & & I \end{bmatrix} \quad \text{and} \quad \mathbb{L} = \begin{bmatrix} I & & \\ & I & \\ -M A^{-1} & & -I \end{bmatrix}.$$

---

<sup>3</sup>Note that in §1 the mapping  $\mu_h$  was considered as  $\mu_h : Q_h^* \rightarrow \mathbb{R}^{n_Q}$ . The definition used here reflects the conceptual distinction between spaces  $\mathbb{R}^{n_Q}$  and  $\mathbb{R}^{n_Q*}$ . That is,  $\mu_h$  is viewed as a map from the space of right-hand sides of the operator equation  $A_h p_h = L_h$  to the space of right-hand sides of the corresponding matrix equation  $A p = b$ .

Here the matrix  $\mathbb{L}$  introduces a Schur complement of a submatrix of  $\mathbb{A}$  corresponding to spaces  $V_h, Q_h$ . The matrix  $\mathbb{U}$  then eliminates the constraint on the space  $U_h$ . Preconditioner  $\mathbb{B}_W$  could now be interpreted as coming from the diagonal of the resulting matrix in (4.7). Further, note that the action of the  $Q_h$ -block can be computed cheaply by Jacobi iterations with a diagonally preconditioned mass matrix (cf. [47]).

Table 4.1: Spectral condition numbers of matrices  $\mathbb{B}_W \mathbb{A}$  for the system assembled on geometry (a) in Figure 3.1.

size	$\log_{10} \epsilon$						
	-3	-2	-1	0	1	2	3
99	2.619	2.627	2.546	3.615	3.998	4.044	4.048
323	2.623	2.653	2.780	3.813	4.023	4.046	4.049
1155	2.631	2.692	3.194	3.925	4.036	4.048	4.049
4355	2.644	2.740	3.533	3.986	4.042	4.048	4.049
16899	2.668	2.788	3.761	4.017	4.046	4.049	4.049
66563	2.703	3.066	3.896	4.033	4.047	4.049	4.049

**4.2. Numerical experiments.** Parameter robust properties of the  $W$ -cap preconditioner are demonstrated by the two numerical experiments used to validate the  $Q$ -cap preconditioner in §3.3. Both the experiments use discretization of domain (a) from Figure 3.1. First, using the *exact* preconditioner we consider the spectral condition numbers of matrices  $\mathbb{B}_W \mathbb{A}$ . Next, using an approximation of  $\mathbb{B}_W$  the linear system  $\mathbb{B}_W \mathbb{A} \mathbf{x} = \mathbb{B}_W \mathbf{f}$  is solved with the minimal residual method. The operator  $\mathbb{B}_W$  is defined as

$$\mathbb{B}_W = \begin{bmatrix} \text{AMG}(\mathbf{A}_U + \epsilon^2 \mathbf{T}^\top \mathbf{A} \mathbf{T}) & & \\ & \text{LU}(\mathbf{A}) & \\ & & \text{LU}(\mathbf{M}) \mathbf{A} \text{LU}(\mathbf{M}) \end{bmatrix}. \quad (4.8)$$

The spectral condition numbers of matrices  $\mathbb{B}_W \mathbb{A}$  for different values of material parameter  $\epsilon$  are listed in Table 4.1. For all the considered discretizations the condition numbers are bounded with respect to  $\epsilon$ . We note that the mesh convergence of the condition numbers appears to be faster and the obtained values are in general smaller than in case of the  $Q$ -cap preconditioner (cf. Table 3.1).

Table 4.2 reports the number of iterations required for convergence of the minimal residual method for the linear system  $\mathbb{B}_W \mathbb{A} \mathbf{x} = \mathbb{B}_W \mathbf{f}$ . Like for the  $Q$ -cap preconditioner the method is started from a random initial vector and the condition  $\mathbf{r}_k^\top \mathbb{B}_W \mathbf{r}_k < 10^{-12}$  is used as a stopping criterion. We find that the iteration counts with the  $W$ -cap preconditioner are again bounded for all the values of the parameter  $\epsilon$ . Consistent with the observations about the spectral condition number, the iteration count is in general smaller than for the system preconditioned with the  $Q$ -cap preconditioner.

We note that the observations from §3.3 about the difference in  $\epsilon$ -dependence of condition numbers and iteration counts of the  $Q$ -cap preconditioner apply to the  $W$ -cap preconditioner as well.

Before addressing the question of computational costs of the proposed preconditioners let us remark that the  $Q$ -cap preconditioner and the  $W$ -cap preconditioners are not spectrally equivalent. Further, both preconditioners yield numerical solutions with linearly (optimally) converging error, see Appendix B.

**5. Computational costs.** We conclude by assessing computational efficiency of the proposed preconditioners. In particular, the setup cost and its relation to the

Table 4.2: Iteration count for system  $\bar{\mathbb{B}}_W \mathbf{A} \mathbf{x} = \bar{\mathbb{B}}_W \mathbf{f}$  solved with the minimal residual method. The problem is assembled on geometry (a) from Figure 3.1. Comparison to the number of iterations with the  $Q$ -cap preconditioned system is shown in the brackets (cf. also Table 3.2).

size	$\log_{10} \epsilon$						
	-3	-2	-1	0	1	2	3
66563	17(-3)	33(-1)	40(3)	30(-2)	20(-8)	14(-10)	12(-9)
264195	19(-3)	35(1)	39(5)	28(-2)	19(-7)	14(-10)	11(-9)
1052675	22(-2)	34(1)	37(5)	27(-1)	19(-7)	14(-8)	11(-7)
4202499	24(-2)	34(2)	34(4)	25(-1)	17(-7)	12(-8)	9(-8)
8398403	25(-1)	32(2)	32(2)	24(-2)	16(-6)	11(-8)	8(-7)
11075583	25(-1)	32(2)	32(2)	25(-1)	16(-6)	13(-6)	11(-4)

aggregate solution time of the Krylov method is of interest. For simplicity we let  $\epsilon = 1$ .

In case of the  $Q$ -cap preconditioner discretized as (3.16) the setup cost is determined by the construction of algebraic multigrid (AMG) and the solution of the generalized eigenvalue problem  $\mathbf{A} \mathbf{x} = \lambda \mathbf{M} \mathbf{x}$  (GEVP). The problem is here solved by calling OpenBLAS[46] implementation of LAPACK[3] routine DSYGVD. The setup cost of the  $W$ -cap preconditioner is dominated by the construction of multigrid for operator  $\mathbf{A}_U + \mathbf{T}^\top \mathbf{A} \mathbf{T}$ . We found that the operator can be assembled with negligible costs and therefore do not report timings of this operation.

The setup costs of the preconditioners obtained on a Linux machine with 16GB RAM and Intel Core i5-2500 CPU clocking at 3.3 GHz are reported in Table 5.1. We remark that timings on the finest discretization deviate from the trend set by the predecessors. This is due to SWAP memory being required to complete the operations and the case should therefore be omitted from the discussion. On the remaining discretizations the following observations can be made: (i) the solution time always dominates the construction time by a factor 5.5 for  $W$ -cap and 3.5 for  $Q$ -cap, (ii)  $W$ -cap preconditioner is close to two times cheaper to construct than the  $Q$ -cap preconditioner in the form (3.16), (iii) the eigenvalue problem always takes fewer seconds to solve than the construction of multigrid.

For our problems of about 11 million nodes in the  $2d$  domain, the strategy of solving the generalized eigenvalue problem using a standard LAPACK routine provided an adequate solution. However, the DSYGVD routine appears to be nearly cubic in complexity ( $\mathcal{O}(n_Q^{2.70})$  or  $\mathcal{O}(n_U^{1.35})$ , cf. Table 5.1), which may represent a bottleneck for larger problems. However, the transformation  $\mathbf{M}_l^{\frac{1}{2}} \mathbf{A} \mathbf{M}_l^{\frac{1}{2}}$  with  $\mathbf{M}_l$  the lumped mass matrix presents a simple trick providing significant speed-up. In fact, the resulting eigenvalue problem is symmetric and tridiagonal and can be solved with fast algorithms of nearly quadratic complexity [20, 21]. Note that the tridiagonal property holds under the assumption of  $\Gamma$  having no bifurcations and that the elements are linear. To illustrate the potential gains with mass lumping, using the transformation and applying the dedicated LAPACK routine DSTEGR we were able to compute eigenpairs for systems of order sixteen thousand in about fifty seconds. This presents more than a factor ten speed up relative to the original generalized eigenvalue problem. The value should also be viewed in the light of the fact that the relevant space  $U_h$  has in this case about quarter billion degrees of freedom. We remark that [28] presents a method for computing all the eigenpairs of the generalized symmetric tridiagonal

eigenvalue problem with an estimated quadratic complexity.

Let us briefly mention a few alternative methods for realizing the mapping between fractional Sobolev spaces needed by the  $Q$ -cap preconditioner. The methods have a common feature of computing the action of operators rather than constructing the operators themselves. Taking advantage of the fact that  $H(s) = MS^{-s}$ ,  $S = A^{-1}M$ , the action of the powers of the matrix  $S$  is efficiently computable by contour integrals [25], symmetric Lanczos process [4, 5] or, in case the matrices  $A$ ,  $M$  are structured, by fast Fourier transform [39]. Alternatively, the mapping can be realized by the BPX preconditioner [12, 11] or integral operator based preconditioners, e.g. [43]. The above mentioned techniques are all less than  $\mathcal{O}(n_Q^2)$  in complexity.

In summary, for linear elements and geometrical configurations where  $\Gamma$  is free of bifurcations the eigenvalue problem required for (2.5) lends itself to solution methods with complexity nearing that of the multigrid construction. In such case the  $Q$ -cap preconditioner (3.16) is feasible whenever the methods deliver acceptable performance ( $n_Q \sim 10^4$ ). For larger spaces  $Q_h$  a practical realization of the  $Q$ -cap preconditioner could be achieved by one of the listed alternatives.

Table 5.1: Timings of elements of construction of the  $Q$ ,  $W$ -cap for  $\epsilon = 1$  and discretizations from Table 3.2, 4.2. Estimated complexity of computing quantity  $v$  at  $i$ -th row,  $r_i = \log v_i - \log v_{i-1} / \log m_i - \log m_{i-1}$  is shown in the brackets. Fitted complexity of computing  $v$ ,  $\mathcal{O}(n_Q^{r_i})$  is obtained by least-squares. All fits but GEVP ignore the SWAP effected final discretization.

$n_U$	$n_Q$	$Q$ -cap			$W$ -cap	
		AMG[s]	GEVP[s]	MinRes[s]	AMG[s]	MinRes[s]
66049	257	0.075(1.98)	0.014(1.81)	0.579(1.69)	0.078(1.94)	0.514(1.73)
263169	513	0.299(2.01)	0.066(2.27)	2.286(1.99)	0.309(1.99)	2.019(1.98)
1050625	1025	1.201(2.01)	0.477(2.87)	8.032(1.82)	1.228(1.99)	7.909(1.97)
4198401	2049	4.983(2.05)	3.311(2.80)	30.81(1.94)	4.930(2.01)	30.31(1.94)
8392609	2897	9.686(1.92)	8.384(2.68)	62.67(2.05)	10.64(2.22)	59.13(1.93)
11068929	3327	15.94(3.60)	12.25(2.74)	84.43(2.15)	15.65(2.79)	82.13(2.37)
Fitted complexity		(2.02)	(2.70)	(1.92)	(2.02)	(1.96)

**6. Conclusions.** We have studied preconditioning of model multiphysics problem (1.1) with  $\Gamma$  being the subdomain of  $\Omega$  having codimension one. Using operator preconditioning [35] two robust preconditioners were proposed and analyzed. Theoretical findings obtained in the present treatise about robustness of preconditioners with respect to material and discretization parameter were demonstrated by numerical experiments using a stable finite element approximation for the related saddle point problem developed herein. Computational efficiency of the preconditioners was assessed revealing that the  $W$ -cap preconditioner is more practical. The  $Q$ -cap preconditioner with discretization based on eigenvalue factorization is efficient for smaller problems and its application to large scale computing possibly requires different means of realizing the mapping between the fractional Sobolev spaces.

Possible future work based on the presented ideas includes extending the preconditioners to problems coupling  $3d$  and  $1d$  domains, problems with multiple disjoint subdomains and problems describing different physics on the coupled domains. In addition, a finite element discretization of the problem, which avoids the constraint for  $\Gamma_h$  to be aligned with facets of  $\Omega_h$  is of general interest.

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### Appendix A. Brezzi theory.

THEOREM A.1 (Brezzi). *The operator  $\mathcal{A} : V \times Q \rightarrow V^* \times Q^*$  in (2.13) is an isomorphism if the following conditions are satisfied*  
 (a) *A is bounded,*

$$\sup_{u \in V} \sup_{v \in V} \frac{\langle Au, v \rangle}{\|u\|_V \|v\|_V} = c_A \equiv \|A\| < \infty, \quad (\text{A.1a})$$

(b) *A is invertible on  $\ker B$ , with*

$$\inf_{u \in \ker B} \frac{\langle Au, u \rangle}{\|u\|_V^2} \geq \alpha > 0 \quad (\text{A.1b})$$

(c) *B is bounded,*

$$\sup_{q \in Q} \sup_{v \in V} \frac{\langle Bv, q \rangle}{\|v\|_V \|q\|_Q} = c_B \equiv \|B\| < \infty, \quad (\text{A.1c})$$

(d) *B is surjective (also inf-sup or LBB condition), with*

$$\inf_{q \in Q} \sup_{v \in V} \frac{\langle Bv, q \rangle}{\|v\|_V \|q\|_Q} \geq \beta > 0. \quad (\text{A.1d})$$

The operator norms  $\|A\|$  and  $\|\mathcal{A}^{-1}\|$  are bounded in terms of the constants appearing in (a)-(d).

*Proof.* See for example [14].  $\square$

**Appendix B. Estimated order of convergence.** Refinements of a uniform discretization of geometry (a) in Figure 3.1 are used to establish order of convergence of numerical solutions of a manufactured problem obtained using  $Q$ -cap and  $W$ -cap preconditioners. The error of discrete solutions  $u_h$  and  $v_h$  is interpolated by discontinuous piecewise cubic polynomials and measured in the  $H_0^1$  norm. The observed convergence rate is linear(optimal).

size	Q-cap		W-cap	
	$ u - u_h _{1,\Omega}$	$ v - v_h _{1,\Gamma}$	$ u - u_h _{1,\Omega}$	$ v - v_h _{1,\Gamma}$
16899	$3.76 \times 10^{-2}(1.00)$	$1.32 \times 10^{-2}(1.00)$	$3.76 \times 10^{-2}(1.00)$	$1.32 \times 10^{-2}(1.00)$
66563	$1.88 \times 10^{-2}(1.00)$	$6.58 \times 10^{-3}(1.00)$	$1.88 \times 10^{-2}(1.00)$	$6.58 \times 10^{-3}(1.00)$
264195	$9.39 \times 10^{-3}(1.00)$	$3.29 \times 10^{-3}(1.00)$	$9.39 \times 10^{-3}(1.00)$	$3.29 \times 10^{-3}(1.00)$
1052675	$4.70 \times 10^{-3}(1.00)$	$1.64 \times 10^{-3}(1.00)$	$4.70 \times 10^{-3}(1.00)$	$1.64 \times 10^{-3}(1.00)$
4202499	$2.35 \times 10^{-3}(1.00)$	$8.22 \times 10^{-4}(1.00)$	$2.35 \times 10^{-3}(1.00)$	$8.22 \times 10^{-4}(1.00)$

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